The Planetary Boundary Layer

The *Planetary Boundary Layer* is the portion of the atmosphere in which the flow is strongly influenced by interaction with the surface of the Earth. Ultimately this interaction depends on molecular viscosity (?). Within the few millimeters of the surface, vertical shears are very intense, and in this layer of the atmosphere molecular diffusion (?) is comparable to other forces in the momentum equation. Outside of this *viscous sublayer* molecular viscosity is a very slow process and is not important in the boundary layer equations for the mean wind, although it is still important for small–scale turbulent eddies. Viscosity's role is indirect but important. It causes the velocity to vanish at the surface, the *no-slip boundary condition*, and even a fairly weak wind will cause a large-velocity shear near the surface which continually leads to the development of turbulent eddies. These turbulent motions have spatial and temporal scales much smaller than those resolved by the meteorological observing network. The source of the turbulence is both strong vertical wind shears and convection. The shear–induced eddies, together with convective eddies caused by surface heating, are very effective in transferring momentum to the surface and transferring heat (latent and sensible) away from the surface at rates many orders of magnitude larger than can be done by molecular processes. The depth of the PBL produced by this turbulent transport may range from as little as 30 m in conditions of large static stability to more than 3 km in highly convective conditions. On average, at mid–latitudes, the PBL extends through the lowest $\sim$ one kilometre of the ground, and contains $\sim$ 10% of the mass of the atmosphere.

The dynamical structure of the PBL is not produced directly by viscosity, but by the fact that in this layer the atmospheric flow is turbulent. In the *free atmosphere* (i.e., the region above the PBL), this (sub-grid scale) turbulence is comparatively small and can be ignored in the approximate treatment of synoptic–scale motions, except perhaps in the vicinity of jet streams, fronts, and convective clouds. However in the PBL we need to modify the dynamical equations to represent the effects of turbulence associated with PBL evolution.

An aside: What is viscosity? It is friction for fluids, and it is another example of a diffusion process. No wind, then molecules move randomly in all directions; a wind, then molecules have a definite tendency to move in a particular direction. Viscosity (fluid friction) is felt when wind is stronger in one place than another (i.e., when there is a wind <u>shear</u>). Then molecules from faster wind regions gradually spread into slower regions and vice versa — result being winds tend to average out. Velocity differences even out more rapidly in a sticky, viscous fluid where molecules are "freer" to interchange.

• Friction almost always acts to slow down moving objects — and the acceleration produced is not given by any simple formula.

• But friction can speed up air and it almost always speeds up water — many of world's surface ocean currents are dragged/driven by friction of the winds at the surface.

• Friction acts to slow winds near any solid boundary.

• At ground, friction always acts in direction opposite from the wind.

• Friction tends to reduce velocity <u>differences</u> throughout the fluid.

This region of the atmosphere (lowest km) where strong turbulent activity occurs is called the planetary boundary layer (PBL).

The source of the turbulence is both strong vertical wind shears and convection.

We assume that turbulence mixing behaves in an analogous manner to molecular mixing. This assumption allows us to determine mathematically the wind structure in the PBL.

The resulting solution, the so-called Ekman Spiral, although not very accurate, does yield a very good <u>qualitative</u> understanding of the effects of turbulent mixing in the PBL.

5.1 ATMOSPHERIC TURBULENCE

Turbulent flow contains irregular quasi–random motions spanning a continuous spectrum of spatial and temporal scales. Such turbulent flows or *eddies* cause air parcels to move so that properties such as momentum and potential temperature are mixed across the boundary layer. Unlike the large-scale rotational flows discussed in Dynamic Meteorology (and in Chapters 3 and 4, Holton 4th Edition) which have depth scales small compared to their horizontal scales, the turbulent eddies in the boundary layer tend to have similar scales in the horizontal and vertical. The maximum eddy length scale is limited by the boundary layer depth and is approximately 10^3 m. The minimum eddy length scale is that of the smallest eddies that can exist in the presence of diffusion by molecular friction.

5.1.1 The Boussinesq Approximation

In the standard (basic state) atmosphere, density varies by 10% across the lowest kilometre of the atmoshere, and perturbations of density from the basic state vary by a few percentage points. Despite these apparently small density variations, density fluctuations are essential for representing the buoyancy force and cannot be ignored totally. To develop dynamical equations for application in the boundary layer from a theoretical point of view, we make some important simplications in the dynamical equations known as the *Boussinesq approximation.* In this approximation, density is replaced by a constant mean value, ρ_0, every where *except in the buoyancy term* in the vertical momentum equation. The horizontal momentum equations are expressed as

$$\frac{Du}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + fv + F_{rx}, \tag{5.1}$$

$$\frac{Dv}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu + F_{ry}, \tag{5.2}$$

and

$$\frac{Dw}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g\frac{\theta}{\theta_0} + F_{ry} \tag{5.3}$$

Here θ is the departure of potential temperature from its basic state value $\theta_0(z)$. The total potential temperature field is given by

$$\theta_{tot} = \theta(x, y, z, t) + \theta_0(z)$$

and the adiabatic thermodynamic energy equation is

$$\frac{D\theta}{Dt} = -w\frac{d\theta_0}{dz}. \tag{5.4}$$

The continuity equation under the Boussinesq approximation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5.5}$$

5.1.2 Reynolds Averaging

In a turbulent fluid, a field variable such as velocity, or temperature, or moisture, or density, measured at a point generally fluctuates rapidly in time as eddies of various scales pass the point (Fig 1.3, Stull's 1988 text). For the measurements to be truly representative of the large-scale flow, an average over an interval of time long enough to average out small–scale eddy flucutations [denoted by $(\ )'$], but short enough to preserve trends in the large–scale flow field [denoted by $\overline{(\ )}$] is necessary. This is called 'Reynolds Averaging.'

Following the scheme introduced by Reynolds, a field variable α can be written as

$$\alpha = \overline{\alpha} + \alpha'$$

where $\overline{\alpha}$ is a running mean (or running time-average)

$$\overline{\alpha} = \frac{1}{\Delta t}\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \alpha(x, y, z, t)dt.$$

Here Δt is chosen so that it is long enough to average out the short term fluctuations, but short enough to retain the long term fluctuations. $\overline{\alpha}$ is α averaged over time interval Δt. Thus $\overline{\alpha}$ varies slowly with time, whereas the high frequency fluctuations removed by the running mean are represented by α'.

We generally associate the slowly-varying quantities as corresponding to the synoptic-scale, whereas the eddy stresses (or superimposed turbulence) are due to small-scale processes. This distinction between a mean flow and superimposed turbulence is justified by the existence of a *spectral gap*, which means that there is only little wind energy on time scales between about 10 minutes (turbulence scales) and 10 hours (synoptic scales).

It is important to note that

$$\overline{\alpha'} = \overline{\alpha} - \overline{\overline{\alpha}} = 0$$

because

$$\overline{\overline{\alpha}} = \overline{\alpha}.$$

The terms which include products of field variables, or velocities in this example, are written and simplified as

$$\begin{aligned}\overline{u\,v} = \overline{(\overline{u} + u')(\overline{v} + v')} =& \overline{u}\overline{v} + \overline{u}\,\underbrace{\overline{v'}}_{=zero} + \underbrace{\overline{v'}}_{=zero}\,\overline{u} + \overline{u'v'} \\ =& \overline{u}\overline{v} + \overline{u'v'}\end{aligned}$$

We can also write

$$\overline{w'\overline{\theta}} = \overline{w'}\overline{\theta} = 0,$$

and

$$\overline{w\,\theta} = \overline{(\overline{w} + w')(\overline{\theta} + \theta')} = \overline{w}\,\overline{\theta} + \overline{w'\theta'}.$$

These equations illustrate how the slowly varying velocities, $\overline{u}$ and $\overline{v}$ depend on the average of the product of the deviation components, i.e., $\overline{u'u'}$, $\overline{u'v'}$, etc. which are the turbulent fluctuations or *covariances*, and represent *eddy stresses*. [See Stull's 1988 text, page 42.]

If for example, on average the turbulent vertical velocity is upward (downward) where the potential temperature deviation is positive (negative), the product $\overline{w'\theta'}$ is positive and the variables are said to be positively *correlated*.

Before applying Reynolds decomposition to (5.1)-(5.4), we rewrite the total derivative in each equation in flux form. We combine the horizontal momentum equations

$$\rho \times \Big[\frac{\partial u}{\partial t} + \vec{V}\cdot\nabla u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\Big],$$

$$\rho \times \Big[\frac{\partial v}{\partial t} + \vec{V}\cdot\nabla v + fv = -\frac{1}{\rho}\frac{\partial p}{\partial y}\Big],$$

with the continuity equation

$$u \text{ or } v \times \Big[\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0\Big],$$

to get the <u>flux</u> form of the momentum equations:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u\,v) + \frac{\partial}{\partial z}(\rho u\,w) - f\rho v = -\frac{\partial p}{\partial x}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial y}(\rho u\,v) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho v w) + f\rho u = -\frac{\partial p}{\partial y}$$

If we separate each dependent variable into mean and fluctuating parts

$$\vec{u} = \overline{u} + u', \qquad \vec{v} = \overline{v} + v',$$

$$\vec{w} = \overline{w} + w', \qquad \vec{p} = \overline{p} + p',$$

where it is assumed that ρ'=0, so that $\rho = \overline{\rho}$, and then time average the flux forms of the momentum equation, we get

$$\frac{\partial}{\partial t}\overline{u} + \overline{u}\frac{\partial}{\partial x}\overline{u} + \overline{v}\frac{\partial}{\partial y}\overline{u} + \overline{w}\frac{\partial}{\partial z}\overline{u} - f\overline{v} =$$

$$-\frac{1}{\rho}\frac{\partial}{\partial x}\overline{p} - \frac{1}{\rho}\Big[\underbrace{\frac{\partial}{\partial x}(\rho\overline{u'u'})}_{eddy\ stress} + \underbrace{\frac{\partial}{\partial y}(\rho\overline{u'v'})}_{eddy\ stress} + \underbrace{\frac{\partial}{\partial z}(\rho\overline{u'w'})}_{eddy\ stress}\Big]$$

$$\frac{\partial}{\partial t}\overline{v} + \overline{u}\frac{\partial}{\partial x}\overline{v} + \overline{v}\frac{\partial}{\partial y}\overline{v} + \overline{w}\frac{\partial}{\partial z}\overline{v} + f\overline{u} =$$

$$-\frac{1}{\rho}\frac{\partial}{\partial y}\overline{p} - \frac{1}{\rho}\Big[\frac{\partial}{\partial x}(\rho\overline{u'v'}) + \frac{\partial}{\partial y}(\rho\overline{v'v'}) + \frac{\partial}{\partial z}(\rho\overline{v'w'})\Big]$$

where the eddy is a primed ($'$) quantity, the stress is a <u>sheared</u> quantity. If you have trouble understanding "flux", then look at the units of flux;

$$\rho u \rightarrow \frac{\mathrm{kg}}{\mathrm{l}^3}\,\frac{\mathrm{l}}{\mathrm{s}} = \frac{\mathrm{kg}}{\mathrm{l}^2}\,\frac{1}{\mathrm{s}},$$

or the amount of mass flowing through a square area in some unit of time.

There are other ways of determining the flux form of the governing equations. For example, the term on the left in (5.1) can be manipulated with the aid of the continuity equation (5.5) and the chain rule of differentiation to yield

$$\begin{aligned}\frac{Du}{Dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big)\\ &= \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial u\,v}{\partial y} + \frac{\partial u\,w}{\partial z}.\end{aligned} \tag{5.6}$$

$$\overline{\frac{Du}{Dt}} = \frac{\partial\overline{u}}{\partial t} + \frac{\partial}{\partial x}\Big(\overline{u}\,\overline{u} + \overline{u'u'}\Big) + \frac{\partial}{\partial y}\Big(\overline{u}\,\overline{v} + \overline{u'v'}\Big) + \frac{\partial}{\partial z}\Big(\overline{u}\,\overline{w} + \overline{u'w'}\Big) \tag{5.7}$$

$$\overline{\frac{Du}{Dt}} = \frac{\overline{D}\overline{u}}{dt} + \frac{\partial}{\partial x}\Big(\overline{u'u'}\Big) + \frac{\partial}{\partial y}\Big(\overline{u'v'}\Big) + \frac{\partial}{\partial z}\Big(\overline{u'w'}\Big) \tag{5.8}$$

where

$$\frac{\overline{D}}{dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z}$$

is the rate of change following the mean motion.

If a further assumption of constant density is made, where $\rho = \rho_0$, then the mean equations are finally

$$\frac{\overline{D}\overline{u}}{dt} = -\frac{1}{\rho_0}\frac{\partial\overline{p}}{\partial x} + f\overline{v} - \Big[\frac{\partial\overline{u'u'}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}\Big] + F_{rx}, \tag{5.9}$$

$$\frac{\overline{D}\overline{v}}{dt} = -\frac{1}{\rho_0}\frac{\partial\overline{p}}{\partial y} - f\overline{u} - \Big[\frac{\partial\overline{u'v'}}{\partial x} + \frac{\partial\overline{v'v'}}{\partial y} + \frac{\partial\overline{v'w'}}{\partial z}\Big] + F_{ry}, \tag{5.10}$$

$$\frac{\overline{D}\overline{w}}{dt} = -\frac{1}{\rho_0}\frac{\partial\overline{p}}{\partial z} + g\frac{\overline{\theta}}{\theta_0} - \Big[\frac{\partial\overline{u'w'}}{\partial x} + \frac{\partial\overline{v'w'}}{\partial y} + \frac{\partial\overline{w'w'}}{\partial z}\Big] + F_{rz}, \tag{5.11}$$

$$\frac{\overline{D}\overline{\theta}}{dt} = -w\frac{d\theta_0}{dz} - \Big[\frac{\partial\overline{u'\theta'}}{\partial x} + \frac{\partial\overline{v'\theta'}}{\partial y} + \frac{\partial\overline{w'\theta'}}{\partial z}\Big], \tag{5.12}$$

$$\frac{\partial\overline{u}}{\partial x} + \frac{\partial\overline{v}}{\partial y} + \frac{\partial\overline{w}}{\partial z} = 0. \tag{5.13}$$

The covariance terms in the square brackets in (5.9)-(5.12) represent turbulent fluxes or *eddy stresses*. For example $\overline{w'\theta'}$ is the turbulent heat flux (or just heat flux) in kinematic form. Similarly $\overline{w'u'} = \overline{u'w'}$ is the vertical turbulent flux of zonal momentum.

Outside the boundary layer, the turbulent fluxes are often sufficiently weak so that the terms in the square brackets in (5.9)-(5.12) can be neglected in the analysis of large–scale (i.e., synoptic scale) flows. This assumption was implicitly made in Holtons 4th Edition, Chapters 3 and 4.

But for many boundary layers, the magnitude of the turbulence flux divergence (?) terms are of the same order as the other terms in (5.9)-(5.12) and cannot be neglected, even when only the mean flow is of direct interest.

The complete equations for the mean flow (5.9)-(5.13), unlike the equations for the total flow (5.1)-(5.5), and the approximate equations of Chapters 3 and 4 are not a closed set. In addition to the five unknown mean variables ($\overline{u}$, $\overline{v}$, $\overline{w}$, $\overline{\theta}$, $\overline{p}$), are unknown turbulent fluxes. This system of equations has more unknowns that equations, and therefore does not form a closed set of equations. To solve this system of equations, *closure* assumptions must be made to approximate the unknown fluxes in terms of the five known mean state variables.

In general the eddy stresses are poorly known and so they are parameterized, i.e., they are written in terms of better known (e.g., measurable or calculated) quantities such as $\overline{u}$, $\overline{v}$.

5.2 TURBULENT KINETIC ENERGY

Vortex stretching and twisting associated with turbulent eddies always tend to cause turbulent energy to flow toward the smallest scales. where it is (finally) dissipated by viscous diffusion (and negligible heating). Thus there must be a continual production of turbulence if the turbulent kinetic energy is to remain statistically stable (?). The primary source of boundary layer turbulence depends on the structure of the wind and the temperature profiles near the surface. If the lapse rate is unstable, boundary layer turbulence is convectively generated. If it is stable, convective is surpressed, and the instability associated with wind shear is responsible for generating turbulence in the boundary layer. [Turbulence is typically generated by a combination of wind shear and near-surface atmospheric instability.] The comparative roles of these processes and the production of turbulence can be understood best by examining the budget for the turbulent kinetic energy.

Turbulent kinetic energy per unit mass is defined as

$$TKE \equiv \frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2}.$$

To determine the budget for TKE, we subtract the component mean momentum equations (5.9)-(5.11) from the corresponding unaveraged equations (5.1)-(5.2), multiple the results by u', v', w', respectively, add the resulting three equations, and average to obtain

$$\frac{\overline{D}(TKE)}{Dt} = MP + BPL + TR - \varepsilon, \tag{5.14}$$

Here MP is the mechanical production, BPL the buoyant production or loss, TR the redistribution by transport and pressure forces, and ε the frictional dissipation. The ε is always positive, reflecting the dissipation of the smallest scales of turbulence by molecular viscosity.

The buoyancy term BPL represents the conversion of energy between mean flow potential energy and turbulence kinetic energy and is defined by

$$BPL \equiv \overline{w'\,\theta'}\left(\frac{g}{\theta_0}\right).$$

If water vapor is present then θ_0 is replaced by virtual potential temperature θ_{0_v}. BPL It is positive for motions that lower the center of mass of the atmosphere, and negative for motions that raise it. Positive buoyancy production occurs when there is heating at the surface so that an unstable temperature lapse rate develops near the ground and spontaneous convective overturning can occur. [See Problems 1.14 and 1.15, Holton 4th Edition in Holton.] Fig 5.1 shows convective eddies have positively correlated vertical velocity and potential temperature fluctuations and hence provide a source of turbulent kinetic energy and positive heat flux. This source of TKE is positive (negative) in a convectively unstable (statically stable) atmosphere and tends to generate (reduce or eliminate) turbulence.

The mechanical production term MP represents a conversion of energy between mean flow and turbulent fluctuations. This term is proportional to the shear in the mean flow and is defined by

$$MP \equiv -\overline{u'w'}\,\frac{\partial \overline{u}}{\partial z} - \overline{v'w'}\,\frac{\partial \overline{v}}{\partial z}. \tag{5.15}$$

For both statically stable and unstable boundary layers, turbulence can be generated mechanically by dynamical instability due to wind shear. MP is positive when the momentum flux is directed down the gradient of the mean momentum. This means that if the mean vertical shear in a layer is westerly ($\partial\overline{u}/\partial z > 0$) then $\overline{u'w'} < 0$ for $MP > 0$.

In a statically stable layer, turbulence can exist only if mechanical production is large enough to overcome the damping effects of stability and viscous dissipation. This is measured by the flux Richardson number which is defined by

$$R_f \equiv -\frac{BPL}{MP}.$$

If the boundary layer is statically unstable, then $R_f < 0$ and turbulence is sustained by convection. If the boundary layer is statically stable, R_f will be positive. Observations suggest that only when R_f is less than 0.25 (i.e., MP exceeds BPL or buoyancy damping by a factor of 4) is the mechanical production intense enough to sustain turbulence in a stable layer. Since MP depends on shear, it always becomes large close enough to the surface. However, as static stability increases, the depth of the layer in which there is a net production of turbulence shrinks. When there is a strong temperature inversion, such as produced by nocturnal radiative cooling at the surface, the boundary layer depth may be on a few decameters and vertical mixing is strongly suppressed. Because R_f involves turbulent correlations, we can use it to determine whether turbulent flow will become laminar, not whether laminar flow will become turbulent.

The TR term represent the sum of the vertical turbulent flux of TKE and of pressure, and is written

$$TR \equiv -\frac{\partial(\overline{w'\,TKE})}{\partial z} - \rho_0\frac{\partial(\overline{w'\,p})}{\partial z}.$$

TR acts on a local scale as either a production or loss depending on whether there is a flux convergence or divergence. When integrated over the depth of the mixed layer (ML below), this terms become identically zero, assuming as bottom and top boundary conditions that the atmosphere is not turbulent at the surface or above the top of the ML.

5.3 PLANETARY BOUNDARY LAYER MOMENTUM EQUATIONS

For the special case of horizontallly homogeneous turbulence above the viscous sublayer (?), molecular viscosity and horizontal turbulent momentum flux divergence terms can be neglected. The mean flow momentum equations (5.9) and (5.10) become

$$\frac{\overline{D}\overline{u}}{Dt} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} + f\overline{v} - \frac{\partial \overline{u'w'}}{\partial z} \tag{5.16}$$

and

$$\frac{\overline{D}\overline{v}}{Dt} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial y} - f\overline{u} - -\frac{\partial \overline{v'w'}}{\partial z}. \tag{5.17}$$

These equations can only be solved for $\overline{u}$ and $\overline{v}$ if the vertical distribution of the turbulent momentum flux is known. Because this depends on the structure of the turbulence, no general solution is possible. A number of semi–empirical methods are used.

For midlatitude synoptic-scale motions, Section 2.4 in Holton 4th Edition showed that to a first approximation the inertial acceleration terms (terms on the left of 5.16 and 5.17) can be neglected compared to other terms in these expressions. Outside the boundary layer the resulting approximation was simply geostrophic balance. In the boundary layer the turbulent terms must be included, and the resulting approximation is

$$f(\overline{v} - \overline{v}_g) - \frac{\partial \overline{u'w'}}{\partial z} = 0, \tag{5.18}$$

$$-f(\overline{u} - \overline{u}_g) - \frac{\partial \overline{v'w'}}{\partial z} = 0 \tag{5.19}$$

a three-way balance between the Coriolis force, the pressure gradient force, and the turbulent momentum flux divergence. Here

$$\mathbf{V}_g \equiv \mathbf{k} \times \frac{1}{\rho f}\nabla p \tag{2.23}$$

is used to express the pressure gradient force in terms of geostrophic velocity.

5.3.1 Well–Mixed Boundary Layer

If a convective boundary layer is topped by a stable layer, turbulent mixing can lead to the formation of a well–mixed (ML) layer. Such boundary layers are common over land during the day when surface heating is strong and over oceans when the air near the sea surface is colder than the surface water temperature.

Observations show that in a well–mixed layer, the wind speed and potential temperature are nearly independent of height (Fig 5.2, Holton 4th Edition; adapted from Stull's text 1988). To a first approximation it is possible to treat the layer as a slab in which the velocity and potential temperature profiles are constant with height and turbulent fluxes vary linearly with height. For simplicity we assume that turbulence vanishes at the top of the boundary layer. Observations also indicate that the surface momentum flux can be represented by a *bulk aerodynamic formula* where

$$(\overline{u'w'})_s = -C_d\,|\overline{V}|\,\overline{u}$$

and

$$(\overline{v'w'})_s = -C_d\,|\overline{V}|\,\overline{v}.$$

C_d is a nondimensional *drag coefficient*, $|\overline{V}| = (\overline{u}^2 + \overline{v}^2)^{1/2}$, and the subscript s denotes surface values (taken at standard anemometer height). Observations show that $C_d \sim 1.5 \times 10^{-5}$ over oceans, and is several times as large over land.

Using this representation for surface momentum flux, (5.18) and (5.19) can be integrated from the surface to the top of the boundary layer where $z = h$ to give

$$f(\overline{v} - \overline{v}_g) = -\frac{(\overline{u'w'})_s}{h} = \frac{C_d\,|\overline{V}|\,\overline{u}}{h}, \tag{5.20}$$

$$-f(\overline{u} - \overline{u}_g) = -\frac{(\overline{v'w'})_s}{h} = \frac{C_d\,|\overline{V}|\,\overline{v}}{h}. \tag{5.21}$$

By choosing the axes such that $\overline{v}_g = 0$, (5.20) and (5.21) are written

$$\overline{v} = \kappa_s|\overline{\mathbf{V}}|\overline{u}, \qquad \overline{u} = \overline{u}_g - \kappa_s|\overline{\mathbf{V}}|\overline{v} \tag{5.22}$$

where $\kappa_s \equiv C_d/(fh)$. You have already seen this wind behavior in your Synoptic Meteorology class; near the surface winds tend to spiral *into* lower pressure centres, *out of* high pressure centres. In the mixed layer wind speed is less than geostrophic speed, and there is a component of motion directed toward lower pressure (that is, to the left of the geostrophic wind in the Northern Hemisphere and to the right of the geostrophic wind in the Southern Hemisphere) whose magnitude depends on κ_s. If we plug in typical values for $\overline{u}_g$ and κ_s into (5.22), $|\overline{\mathbf{V}}|$ will be less than the magnitude of the geostrophic wind.

So now instead of geostrophic balance, there is a three–way balance between the pressure gradient force, the Coriolis force, and turbulent drag:

$$fk \times \overline{\mathbf{V}} = -\frac{1}{\rho_0}\nabla\overline{p} - \frac{C_d}{h}|\overline{\mathbf{V}}|\overline{\mathbf{V}}. \tag{5.23}$$

This balance of forces is illustrated in Fig 5.3 [Holton 4th Edition]. The effects of this extra frictional drag on the momentum are: (1) to slow the flow; (2) to cause the flow to turn somewhat to the left (right) in the NH (SH).

The Coriolis force acts perpendicular to the motion, the turbulent drag is a retarding force, and their sum must balance the pressure gradient force that is directed toward lower pressure. As turbulent drag increases, the cross isobar angle increases.

The geostrophic wind flows parallel to the isobars without frictional drag; switch on drag and the wind suddenly slows. As it slows down, the Coriolis force $(fv, -fu)$ also decreases. This destroys the geostrophic balance, wind accelerates (i.e., turns) toward its left, into low pressure.

Winds spiral near ground into Lows and out of Highs largely because of frictional drag. [Look at the maps in Synoptic Metoeorology and closed low pressure centres over the Rockies. How are the surface winds directed around these Lows?]

The work done by the flow toward lower pressure that balanced the frictional dissipation at the surface. Because BL turbulence tends to reduce wind speeds, the turbulent momentum fluxes are often referred to as bl *friction*. Note that the forces involved are due to turbulence, not molecular viscosity.

5.3.2 The Flux–Gradient Theory

$$(\overline{u'w'}) = -K_m\Big(\frac{\partial \overline{u}}{\partial z}\Big),$$

$$(\overline{v'w'}) = -K_m\Big(\frac{\partial \overline{v}}{\partial z}\Big),$$

and

$$(\overline{\theta'w'}) = -K_h\Big(\frac{\partial \overline{\theta}}{\partial z}\Big),$$

where K_m [units m^2 s^{-1}] is the *eddy viscosity* (?) coefficient and K_h is the *eddy diffusivity* (?) of heat. This *closure scheme* is referred to as the *K theory*.

Eddy viscosity, unlike molecular viscosity, depends on the flow rather than the physical (molecular) properties of the fluid, and must be determined empirically for each situation.

For example, simple models assume that the eddy exchange coefficient is constant throughout the flow, which may be adequate for estimating small-scale *diffusion* of *passive tracers* in the *free* atmosphere. However, this is a poor approximation in the BL where the scales and intensities of typical turbulent eddies are strongly dependent on the distance to the surface (such as in urban settings or sharp terrain, with wake flows, etc) as well as the static stability.

Furthermore, in many atmospheric cases, the most energetic eddies have dimensions comparable to the depth of the boundary layer, and neither the momentum or heat flux is proportional to the local gradient of the mean. A good example of this is in a mixed layer where the heat fluxes can be positive even though the mean stratification in a ML may be very close to neutral (i.e., $\partial\overline{\theta}/\partial z = 0$).

5.3.3 The Mixing Length Hypothesis

The simplest approach for determining a suitable model (i.e., parameterizing) for the eddy diffusion coefficient in the PBL is based on the mixing length hypothesis introduced by fluid dynamicist L. Prandtl. According to this theory we write, for example,

$$u'(z) = \overline{u(z-\Delta z)} - \overline{u(z)} = -\xi' \frac{\partial \overline{u}}{\partial z}.$$

Physically this implies that a parcel moves a characteristic distance ξ' while it keeps its *zonal* velocity $\overline{u}$ fixed. Only the *shear* in the zonal velocity over distance ξ' accounts for the difference $\overline{u(z-\Delta z)} - \overline{u(z)}$. This hypothesis assumes that a parcel of fluid displaced vertically will carry the mean properties of its original level for a characteristic distance ξ and then will mix with its surroundings, just as an average molecule travels a mean free path at constant momentum before colliding and exchanging momentum with another molecule. By further analogy to the molecular mechanism, this displacement is postulated to create a turbulent fluctuation whose magnitude depends on ξ' and the gradient of the mean property. For example

$$\theta' = -\xi' \frac{\partial \overline{\theta}}{\partial z};\; u' = -\xi' \frac{\partial \overline{u}}{\partial z};\; v' = -\xi' \frac{\partial \overline{v}}{\partial z}.$$

The quantity ξ', which is positive for upward parcel displacement, negative for downward, is called the "mixing" length. It is analogous to the mean free path in molecular viscosity where on average a molecule travels a distance equal to its mean free path at constant momentum until it collides with another molecule. In mixing length theory we are assuming that a parcel is displaced a distance ξ' by a turbulent eddy.

For a conservative property such as potential temperature this hypothesis is reasonable provided the eddy scales are small compared to the mean flow scale or that the mean gradient is constant with height. However the hypothesis is less justified in the cast of velocity, as pressure gradient forces may cause substantial changes in the velocity during an eddy displacement.

When mixing is complete (the parcel completely mixes with the air), the eddy term for the vertical turbulent flux of zonal momentum $\rho\overline{u'w'}$ then equals

$$-\rho\overline{u'w'} = \rho\overline{w'\xi'}\frac{\partial \overline{u}}{\partial z}, \tag{5.24}$$

with analogous expressions for the momentum flux in the meridional direction and the potential temperature flux.

In order to estimate w' in terms of the mean fields, we assume that the vertical stability of the atmosphere is nearly neutral (true for flux-gradient or K theory) so that the buoyancy effects are small so that the horizontal scale of the eddies are comparable to the vertical scale and $|w'|$ is similar to $|\mathbf{V}'|$. Therefore we write

$$w' \approx \xi' \left|\frac{\partial \overline{\mathbf{V}}}{\partial z}\right|$$

where $\overline{\mathbf{V}}$ and $\mathbf{V}'$ are the mean and turbulent parts of the horizontal vector velocity field, respectively. The absolute value of $\mathbf{V}$ is needed because if $\xi' > 0$ then $w' > 0$ (i.e., upward parcel displacements are associated with upward eddy velocities.) Thus the momentum flux is

$$-\rho\overline{u'w'} = \rho\overline{(\xi')^2}\left|\frac{\partial \mathbf{V}}{\partial z}\right|\frac{\partial \overline{u}}{\partial z} = \rho\, K_m \frac{\partial \overline{u}}{\partial z}. \tag{5.25}$$

The eddy viscosity is now defined as

$$K_m \equiv \overline{\xi'^2}\left|\frac{\partial \overline{\mathbf{V}}}{\partial z}\right| = \overline{l^2}\left|\frac{\partial \overline{\mathbf{V}}}{\partial z}\right|$$

Here l is the mixing length,

$$l \equiv \left(\overline{\xi'^2}\right)^{1/2},$$

and is the root mean square parcel displacement, which is a measure of eddy size. This result suggests that larger eddies and greater shear induce greater turbulent mixing.

How are these eddies created and what do they do exactly?

Top of PBL (~ 1 km) No Eddies Above

$|l'|$

$|l'|$

ground

Sizes of eddies increase with height since $|l'|$ increase with height, and $|l'|$ — or mean free path — increases with z since density ρ decreases with z.

Parcel trajectory in turbulent eddies created by $\partial\rho/\partial z < 0$ and $\partial u/\partial z$ large.

Nevertheless with this expression for $\rho\overline{u'w'}$ we now have a convenient mathematical representation for the effect of turbulent fluctuations on the synoptic scale. This enables us to develop a theoretical model for the effects of turbulence.

In addition we have

$$-\rho\overline{v'w'} = \rho\overline{(l')^2}\left|\frac{\partial \vec{V}_h}{\partial z}\right|\frac{\partial \overline{v}}{\partial z} = A_z\frac{\partial \overline{v}}{\partial z}.$$

We assume that the exchange coefficient A_z is the same in the zonal and meridional directions — reasonable given that A_z is known to poor accuracy.

Also we only retain the <u>vertical</u> eddy stress terms $\overline{v'w'}$ and $\overline{u'w'}$ because the horizontal derivatives of $\overline{u}$ and $\overline{v}$ are much smaller than their vertical derivatives in the PBL.

Then if we write the eddy stresses as

$$\tau_x = -\rho\overline{u'w'} = A_z\frac{\partial \overline{u}}{\partial z},$$

$$\tau_y = -\rho\overline{v'w'} = A_z\frac{\partial \overline{v}}{\partial z},$$

the horizontal momentum equations become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_x}{\partial z}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_y}{\partial z}$$

where the $\overline{\quad}$ brackets for slowly-varying synoptic-scale terms are omitted. If we assume A_z is constant, then

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \kappa\frac{\partial^2 v}{\partial z^2}$$

where the $\kappa = \frac{A_x}{\rho}$ is the eddy viscosity coefficient.

5.3.4 The Ekman Layer

We saw in the scale analysis for the horizontal momentum equation, that above the PBL where frictional processes are small, the horizontal wind tends to be in geostrophic balance. But because $u, v \to 0$ at the ground, the terms

$$K_m \frac{\partial^2 u}{\partial z^2}, \quad K_m \frac{\partial^2 v}{\partial z^2}$$

(<u>double</u> z derivatives of u, v) must eventually become as large as the pressure gradient and Coriolis forces. Thus as an approximation to the flow in the PBL where dissipative and friction processes are important, the horizontal momentum equations are

$$K_m \frac{\partial^2 u}{\partial z^2} + f(v - v_g) = 0, \tag{5.26}$$

$$K_m \frac{\partial^2 v}{\partial z^2} - f(u - u_g) = 0, \tag{5.27}$$

where u_g and v_g are the geostrophic winds.

Note that near ground (over ocean), the real wind is slowed by 25 to 75 % by friction, and as expected the wind (water) speed and direction changes as the effect of friction changes as you move up (down) into the PBL (ocean).

We can show that the increasing wind speed and decreasing deflection with height results in a wind spiral to the right as you rise through the lowest km of the atmosphere (or friction layer). This is called the Ekman spiral (after Norwegian scientist Vagn Walfrid Ekman who first explained it mathematically).

To do this we need to solve the above *Ekman layer* equations (5.26) and (5.27). To keep the analysis simple, we assume that K_m is constant, and u_g, v_g are also constant, which implies that the density and horizontal pressure gradient are independent of height in the PBL. (i.e., $0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$, $p = \overline{p}(x, y)$.)

Also because we have a total of 4 derivatives in the above system, there must be 4 boundary conditions. These are

$$\begin{aligned} u = 0, \quad v = 0 \text{ at } z\text{=0 (ground)};\\ u \to u_g, \quad v \to v_g \text{ at } z \to \infty \text{ (top of PBL)} \end{aligned} \tag{5.28}$$

since at the top of the PBL the horizontal winds are in geostrophic balance. Friction acts to bring the finite "interior" (i.e., where friction is negligible) velocity gradually to zero at the surface.

To solve the above system, we multiply the meridional momentum equation by $i = \sqrt{-1}$ to give

$$K_m \frac{\partial^2}{\partial z^2}(u + iv) - if(u + iv) = -if(u_g + i \underbrace{v_g}_{=set\ zero})$$

and assume that the geostrophic wind is independent of height, the flow is oriented so that v_g=0, and there is zonal u_g flow only at $z \to \infty$. The general solution to this ODE is

$$u + iv = A \exp\left\{+\sqrt{\frac{if}{\kappa}}z\right\} + B \exp\left\{-\sqrt{\frac{if}{\kappa}}z\right\} + u_g. \tag{5.30}$$

Applying the boundary conditions to solve for coefficients A and B gives for the Northern Hemisphere ($f > 0$)

$$u = u_g(1 - \mathrm{e}^{-\gamma z} \cos \gamma z),$$

$$v = u_g \mathrm{e}^{-\gamma z} \sin \gamma z,$$

where

$$\gamma = \left(\frac{f}{2\,K_m}\right)^{\frac{1}{2}}.$$

Here we have used $\sqrt{i} = (1+i)/\sqrt{2}$, the Euler formula $e^{-i\theta} = \cos\theta - i\sin\theta$, and separated real and imaginary parts. This solution is the famous Ekman spiral expression. There is a flow transverse (i.e., v) to the "interior" flow (where friction is negligible).

Fig 5.4 (Holton 4th Edition) shows a hodograph of the wind components $\left(\frac{u}{u_g}, \frac{v}{u_g}\right)$ for several γz layers in the Ekman spiral.

Problems 5.1*, 5.2*, and 5.3* (and also more difficult mixed layer Problem 5.10) can be done as this point.

When $\gamma z = \pi$ so that $(\cos\pi = -1)$

$$u = u_g(1 + \mathrm{e}^{-\pi}) \sim 1.1 u_g, \text{ and } v = 0,$$

the wind is parallel to and very nearly equal to the geostrophic value. Since u and v are close to their geostrophic values when $\gamma z = \pi$, we can estimate the PBL depth by

$$D_e = \frac{\pi}{\gamma} = \frac{\pi}{(f/2\kappa)^{1/2}} = \left(\frac{2\pi^2\kappa}{f}\right)^{1/2}$$

where D_e is the Ekman layer depth. This shows that the PBL depth: (1) increases with eddy viscosity; (2) decreases with latitude ($f \to 0$ at the equator, PBL depth $\to \infty$!!).

As in the mixed layer case, the Ekman layer solution has a boundary layer wind component directed toward lower pressure, a direct result of the three–way balance between the pressure gradient force, the Coriolis force, and the turbulent drag. As friction decreases or equivalently $\kappa\frac{\partial^2\vec{V}}{\partial z^2}$ decreases, the angle between the wind velocity and the isobars decreases, until finally at the top of the PBL, $u \to u_g$ $v \to v_g$ and the wind blows parallel to the isobars.

The ideal Ekman layer is rarely observed in the atmosphere. Turbulent momentum fluxes are not simply proportional to the gradient of the mean momentum, and in reality K_m is not a constant, and does vary rapidly with height near the ground. The Ekman layer solution should not be carried all the way to the surface.

5.3.5 The Surface Layer

Some of the inadequacies of the Ekman layer model are overcome if we distinguish a *surface layer* from the remainder of the PBL. The surface layer, whose depth depends on stability, is usually less than 10% of the total BL depth (unstable during the day due to in-coming solar radiation and surface heating, stable during the night due to outgoing long-wave IR radiation and surface cooling), and is maintained entirely by vertical momentum transfer by the turbulent eddies. It is not directly dependent on the Coriolis or pressure gradient forces. Supposing that the wind close to the surface is directed parallel to the x axis, the kinematic turbulent momentum flux can then be expressed in terms of

$$u_*^2 \equiv |(\overline{u'\,w'})_s|,$$

where u_* is the *friction velocity.* Observations indicate that the magnitude of the surface momentum flux is ~ 0.1 m^2 s^{-2}. The friction velocity is therefore $\sim .3$ m s^{-1}.

In Section 2.4 (Holton 4th Edition), scale analysis indicates that the Coriolis and pressure gradient force terms in (5.16) are of magnitude $\sim 10^{-3}$ s^{-2} in the midlatitudes. The momentum flux divergence in the surface layer cannot exceed this magnitude or else it would be unbalanced. Therefore

$$\frac{\delta(u_*^2)}{\delta z} \leq 10^{-3} \text{ m s}^{-2}.$$

For δz= 10 m, this implies that

$$\delta(u_*^2) \leq 10^{-2} \text{ m}^2 \text{ s}^{-2}$$

so that the change in the vertical momentum flux in the lowest 10 m of the atmosphere is less than 10% of the surface flux. As a first order approximation we therefore assume that in the lowest several metres of the atmosphere, the turbulent flux remains constant at its surface value, and with (5.25) we write

$$K_m\frac{\partial\overline{u}}{\partial z} = u_*^2. \tag{5.32}$$

We have *parameterized* the surface momentum flux in terms of the eddy viscosity coefficient. In applying K_m in the Ekman layer solution, we assumed a constant value throughout the BL. Near the surface, however, the vertical eddy scale is limited by the distance to the surface. A logical choice for mixing length is

$$l = k\,z,$$

where k is a constant, which leads to

$$K_m = (kz)^2\left|\frac{\partial \overline{u}}{\partial z}\right|.$$

Substituting this into (5.32), and taking the square root of the result gives

$$\frac{\partial \overline{u}}{\partial z} = \frac{u_*}{kz} \tag{5.33}$$

which if integrated with respect to z yields

$$\overline{u} = \frac{u_*}{k}\ln\Big(\frac{z}{z_0}\Big), \tag{5.34}$$

a *logarithmic wind profile*. Here z_0 is the *roughness length*, a constant of integration chosen so that $\overline{u}$=0 at $z = z_0$. The constant k is a universal constant known as *von Karman's constant*, which experimentally determined is $k \approx 0.4$. The roughness height z_0 depends on the physical characteristics of the underlying surface. For example, the z_0 for ice is $\sim 10^{-5}$ m (small), for grassy fields is $\sim 10^{-2}$ m, and for mountains, like the Rocky Mountains, is 100 m. Although a number of simplifying assumptions are required to derive (5.34), observations show that the logarithmic profile is a satisfatory fit to wind profiles in the surface layer.

5.3.6 The Modified Ekman Layer

As stated previously, the classical Ekman layer solution should not be carried all the way to the surface, and a more satisfactory representation of the PLB is obtained by matching the bottom of the Ekman layer with the top of the logarithmic surface layer profile. But, although an improvement, even this modfied Ekman spiral does not represent a good fit to observations. This can be seen in Fig 5.5 (Holton 4th Edition) that shows a mean wind hodograph compared with the modified Ekman spiral solution. Although the detailed structure is rather different from the Ekman spiral, the vertically integrated horizontal mass transport in the BL is still directed toward lower pressure, and this is of primary importance for synoptic and larger scale motions . . . as is discussed next.

We can estimate the <u>vertical</u> <u>velocity</u> at the top of the PBL.

If we assume that $\frac{\partial \rho}{\partial t} = 0$, then the continuity equation can be written

$$\nabla \cdot \rho\vec{V} = 0$$

or

$$\frac{\partial}{\partial z}\rho w = -\Big[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\Big].$$

Integrating through the Ekman layer depth gives

$$\int_0^{D_e}\frac{\partial}{\partial z}(\rho w)dz = -\int_0^{D_e}\Big[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\Big]dz$$

or

$$\rho w(D_e) - \rho\underbrace{w(0)}_{=zero} = -\int \qquad \downarrow$$

or

$$\rho w(D_e) = -\int \qquad \downarrow$$

since $w(0) = 0$.

If we substitute the Ekman layer solutions for u and v (where $v_g = 0$, ζ is assumed constant wrt z), noting that $\nabla \cdot \vec{V}_g = 0$, and therefore

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} = 0$$

(u is independent of x at top of Ekman spiral), we get

$$\rho w(D_e) = -\frac{\partial}{\partial y}\int_0^{D_e} \rho u_g \mathrm{e}^{-\pi z/D_e} \sin\frac{\pi z}{D_e} dz \simeq -\rho\frac{\partial u_g}{\partial y}\left(\frac{\kappa}{2f}\right)^{1/2}$$

or

$$w(\text{at top of Ekman spiral}) = \zeta_g\left(\frac{\kappa}{2f}\right)^{1/2}$$

where $\zeta_g = -\frac{\partial u_g}{\partial y}$.

This vertical velocity in the "interior" is called Ekman pumping. So the greater the vorticity of the mean flow, the greater the upwelling in the atmosphere (downwelling in oceans).

This shows that the vertical velocity at the top of the PBL increases with ζ_g or equivalently

$$\zeta_g = -\frac{\partial u_g}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{\rho f}\frac{\partial p}{\partial y}\right)$$

where $u_g = \frac{1}{\rho f}\frac{\partial p}{\partial y}$.

Examine the diagram.

From the diagram we see that

$$-\frac{\partial u_g}{\partial y} = \zeta_g > 0$$

(since $\frac{\partial^2 p}{\partial y^2} > 0$ for $\zeta_g > 0$, & u_g is therefore decreasing with increasing y).

At $\kappa \neq 0$, $z < D_e$, and some fixed z,

$\frac{v}{u_g}$= constant, or v= constant $\times u_g$

and for u_g changing in y direction, we have

$$\frac{\partial v}{\partial y} \neq 0,$$

and for

$$\zeta_g > 0, \qquad \frac{\partial v}{\partial y} < 0.$$

There is a horizontal (y-direction) convergence of mass therefore ($\frac{\partial v}{\partial y} < 0$), and by mass conservation there must be a positive vertical velocity in the PBL.

The mass flux out of the PBL is given by

$$\rho_0 w(\text{at } D_e) = -\frac{\partial M}{\partial y} \text{ (where } M = \textstyle\int_0^{D_e} \rho_0 v dz)$$

which is equal to the convergence of the cross-isobar mass transport in the layer.

Remember (scale analysis) geostrophic motions must be characterized by

$$\frac{\partial}{\partial z}\overline{w} = 0,$$

i.e., no divergence to geostrophic flow. So (constant wrt height) vertical velocity w must occur through the depth of the fluid.

This means convergence in bottom of Ekman layer and compensating upwelling in the "interior". Such a situation arises in presence of a cyclonic gyre in the "interior".

Anticyclonic gyres in the ocean can therefore "suck" you down.

It is now possible to do Holton 3rd, Problems 5.6, 5.7 (which uses result of 5.6), 5.8, 5.9 (which is spin down of 5.8), and 5.5 (mixed boundary layer).

Ekman or Boundary Layer (BL) Pumping

The above derivations show the horizontal winds in the BL and the vertical velocity at the top of the BL.

We now ask, does the BL flow affects the winds <u>above</u> the BL where turbulent processes are very weak?

We answer this question by using the barotropic vorticity equation (4.24) for synoptic-scale flow:

$$\frac{D}{Dt}(\zeta_g + f) = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = f\frac{\partial w}{\partial z}, \tag{5.39}$$

where we assume that density ρ is constant, ζ_g can be neglected compared to f in the divergence term, and that the latitudinal variation of f can also be neglected. Since the geostrophic winds are independent of height in a constant density, barotropic atmosphere back), the above equation can be integrated from the top of the Ekman layer ($z = D_e$) to the tropopause ($z = H$), giving

$$\int_{z=D_e}^{z=H} \frac{D}{Dt}(\zeta_g + f)dz = \int_{z=D_e}^{z=H} f\frac{\partial w}{\partial z},$$

$$(H - D_e)\frac{D\zeta_g}{Dt} = f[w(H) - w(D_e)] = -fw(D_e)$$

where we assume w(H)=0 and f=constant. Since

$$w(D_e) = \zeta_g\left(\frac{K_m}{2f}\right)^{1/2} \text{ and } H - D_e \sim H, \tag{5.38}$$

the above equation can be written as

$$\frac{D\zeta_g}{Dt} = -\frac{f}{H}\zeta_g\Big(\frac{K_m}{2f}\Big)^{1/2} = -\Big(\frac{fK_m}{2H^2}\Big)^{1/2}\zeta_g.$$

Integrating this equation with respect to time gives

$$\int_0^t \frac{1}{\zeta_g}\frac{D\zeta_g}{Dt}dt = -\int_0^t \Big(\frac{fK_m}{2H^2}\Big)^{\frac{1}{2}}dt,$$

which has the solution

$$\zeta_g = \zeta_g(0)\mathrm{e}^{-\left(\frac{fK_m}{2H^2}\right)^{\frac{1}{2}}t} = \zeta_g(0)\mathrm{e}^{\frac{-t}{\tau_e}},$$

where $\zeta_g(0)$ is the geostrophic vorticity at time $t = 0$

This shows that the vorticity field will decay to zero with an e-folding time (the time is takes a field to decrease to e^{-1} of its original value) of

$$\tau_e = \Big(\frac{2H^2}{f\kappa}\Big)^{-1/2} \sim \text{ 4 days}$$

for typical atmospheric values (H=10 km, f=10^{-4} s^{-1}, κ= 10 m^2 s^{-1}). This e-folding time scale is referred to as the barotropic *spin–down time*, which for midlatitude synoptic–scale disturbances is a few days. Physically this result states that the horizontal convergence or divergence above the BL, which is induced by the flow $w(D_e)$ out the top of the Ekman layer (i.e., Ekman or boundary layer pumping), acts to spin down or destroy vorticity in the atmosphere. Thus turbulent processes in the BL do have a dramatic effect on the synoptic-scale flow above the BL.

Compare this decay time scale to the time scale for ordinary viscous diffusion. This time scale is estimated from a scale analysis of the diffusion equation

$$\frac{\partial u}{\partial t} = K_m\frac{\partial^2 u}{\partial z^2}. \tag{5.43}$$

If τ_d is the diffusive time scale and H is the characteristic vertical scale for diffusion, from the diffusion equation

$$\frac{U}{\tau_d} \sim \frac{K_m U}{H^2},$$

so that

$$\tau_d \sim \frac{H^2}{K_m}.$$

For typical values of H and K_m, the diffusive time scale is about 100 days. In the absence of convective clouds, the spin–down process is far more effective at destroying vorticity in a rotating atmosphere than eddy diffusion. Deep cumulonimbus convection can produce rapid turbulent transports of heat and momentum throughout the entire depth of the atmosphere, and these must be considered together with boundary layer pumping for intense systems such as hurricanes.

You can now do Problems 5.4 and 5.9.

The next question is, how are the winds maintained against the dissipative effects of BL turbulence?

As we will see in Atmospheric Dynamics II, it is the meridional (N-S) temperature gradient set up by the solar differential heating that accomplishes this task.

You can actually see something similar to this happen in a teacup of loose tea, except that the centrifugal force replaces the Coriolis force.

Stir the loose tea so that it spins rapidly — the tea then piles against the wall of your teacup; level of tea is low in the centre, so pressure is low there also.

Velocity drops to zero at the bottom and middle of the cup; therefore we get BL convergence, and by mass conservation, also get upward motion out of the BL.

Stop stirring and tea stops spinning, velocity goes to zero. This is analogous to <u>removing</u> thermal forcing in the atmosphere. At bottom of cup, where there is more friction, tea rapidly stops spinning; the PGF makes tea drift towards centre, and tea leaves pile up, indicating convergence.

"Tea Cup"

Lecture 21

When mixing is complete (the parcel completely mixes with the air), the eddy term $\rho\overline{u'w'}$ then equals

$$-\rho\overline{u'w'} = \rho\overline{w'l'}\frac{\partial\overline{u}}{\partial z}.$$

Also for turbulent eddies, $|w'|$ is similar to $|v'|$. Therefore we can write

$$w' \approx l'\left|\frac{\partial\vec{V}_h}{\partial z}\right|$$

where $\vec{V}_h$ is the horizontal vector velocity field. Thus

$$-\rho\overline{u'w'} = \rho\overline{(l')^2}\left|\frac{\partial\vec{V}_h}{\partial z}\right|\frac{\partial\overline{u}}{\partial z} = A_z\frac{\partial\overline{u}}{\partial z}.$$

The quantity A_z is called the exchange coefficient and its magnitude is poorly known (only within a factor of 10).

Nevertheless with this expression for $\rho\overline{u'w'}$ we now have a convenient mathematical representation for the effect of turbulent fluctuations on the synoptic scale. This enables us to develop a theoretical model for the effects of turbulence.

In addition we have

$$-\rho\overline{v'w'} = \rho\overline{(l')^2}\left|\frac{\partial\vec{V}_h}{\partial z}\right|\frac{\partial\overline{v}}{\partial z} = A_z\frac{\partial\overline{v}}{\partial z}.$$

We assume that the exchange coefficient A_z is the same in the zonal and meridional directions — reasonable given that A_z is known to poor accuracy.

Also we only retain the <u>vertical</u> eddy stress terms $\overline{v'w'}$ and $\overline{u'w'}$ because the horizontal derivatives of $\overline{u}$ and $\overline{v}$ are much smaller than their vertical derivatives in the PBL.

Then if we write the eddy stresses as

$$\tau_x = -\rho\overline{u'w'} = A_z\frac{\partial\overline{u}}{\partial z},$$

$$\tau_y = -\rho\overline{v'w'} = A_z \frac{\partial \overline{v}}{\partial z},$$

the horizontal momentum equations become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_x}{\partial z}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_y}{\partial z}$$

where the $\overline{}$ brackets for slowly-varying synoptic-scale terms are omitted. If we assume A_z is constant, then

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \kappa\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \kappa\frac{\partial^2 v}{\partial z^2}$$

where the $\kappa = \frac{A_x}{\rho}$ is the eddy viscosity coefficient.

As we saw in the scale analysis for the horizontal momentum equation, above the PBL where frictional processes are small, the horizontal wind tends to be in geostrophic balance. But because $u, v \to 0$ at the ground, the terms

$$\kappa\frac{\partial^2 u}{\partial z^2}, \quad \kappa\frac{\partial^2 v}{\partial z^2}$$

(<u>double</u> z derivatives of u, v) must eventually become as large as the pressure gradient and Coriolis forces.

Thus as an approximation to the flow in the PBL where dissipative and friction processes are important, the horizontal momentum equations are

$$\kappa\frac{\partial^2 u}{\partial z^2} + f(v - v_g) = 0,$$

$$\kappa\frac{\partial^2 v}{\partial z^2} - f(u - u_g) = 0,$$

where u_g and v_g are the geostrophic winds.

The effects of this extra frictional drag on the momentum are:

(1) to slow the flow;

(2) to cause the flow to turn somewhat to the L (R) in the NH (SH).

The geostrophic wind flows parallel to the isobars without frictional drag; switch on drag and the wind suddenly slows. As it slows down, the Coriolis force (fv, $-fu$) also decreases. This destroys the geostrophic balance, wind accelerates (i.e., turns) toward its left, into low pressure.

Winds spiral near ground into Lows and out of Highs largely because of frictional drag.

Holton's Fig. 5.3 (p. 126, 3rd Ed)

Note that near ground (over ocean), the real wind is slowed by 25 to 75 % by friction, and as expected the wind (water) speed and direction changes as the effect of friction changes as you move up (down) into the PBL (ocean).

We can show that the increasing wind speed and decreasing reflection with height results in a wind spiral to the right as you rise through the lowest km of the atmosphere (or friction layer).

This is called the Ekman spiral (after Norwegian scientist Vagn Walfrid Ekman who first explained it mathematically).

To do this we need to solve the above momentum equations, and to do this we assume that κ_z is constant, and u_g, v_g are also constant, which implies that the density and horizontal pressure gradient are independent of height in the PBL. (i.e., $0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$, $p = \overline{p}(x, y)$.)

Also because we have a total of 4 derivatives in the above system, there must be 4 boundary conditions. These are

$$u = v = 0 \qquad \text{at } z\text{=0 (ground);}$$

and

$$u = u_g,\ v = v_g \qquad \text{at } z = \infty \text{ (top of PBL)}$$

since at the top of the PBL the horizontal winds are in geostrophic balance.

Friction acts to bring the finite interior (where friction is negligible) velocity gradually to zero at the surface.

To solve the above system, it is convenient to multiply the meridional momentum equation by $i = \sqrt{-1}$ to give

$$\kappa\frac{\partial^2}{\partial z^2}(u + iv) - if(u + iv) = -if(u_g + i\underbrace{v_g}_{=set\ zero})$$

and for simplicity we set v_g=0 (zonal u_g flow only at $z \to \infty$).

The general solution to this ODE is

$$u + iv = A\exp\left\{+\sqrt{\frac{if}{\kappa}}z\right\} + B\exp\left\{-\sqrt{\frac{if}{\kappa}}z\right\} + u_g.$$

Applying the boundary conditions (to solve for coefficients A and B) gives

$$u = u_g(1 - \mathrm{e}^{-\gamma z}\cos\gamma z),$$

$$v = u_g\mathrm{e}^{-\gamma z}\sin\gamma z,$$

where

$$\gamma = \left(\frac{f}{2\kappa}\right)^{\frac{1}{2}}.$$

This solution is called the Ekman spiral expression. There is a flow transverse (i.e., v) to the "interior" flow (where friction is negligible).

Fig. 5.4 (Holton, p. 130, 3rd Ed)

Hodograph arrows show velocity vectors $\left(\frac{u}{u_g}, \frac{v}{u_g}\right)$ for several γz layers in the Ekman spiral.

Problems 5.1*, 5.2*, and 5.3* * (and also more difficult mixed layer Problem 5.10) can be done as this point.

Lecture 22

We can also estimate the <u>depth</u> of the boundary layer.

We first set $\gamma z = \pi$ so that $(\cos\pi = -1)$

$$u = u_g(1 + \mathrm{e}^{-\pi}) \sim 1.1u_g$$

$$v = 0$$

Since u and v are close to their geostrophic values when $\gamma z = \pi$, we can estimate the PBL depth by

$$D_e = \frac{\pi}{\gamma} = \frac{\pi}{(f/2\kappa)^{1/2}} = \left(\frac{2\pi^2\kappa}{f}\right)^{1/2}$$

where D_e is the Ekman layer depth.

This shows that the PBL depth:

1. increases with eddy viscosity;

2. decreases with latitude ($f \to 0$ at the equator, PBL depth $\to \infty$!!).

* Part of Assignment 9: 5.1, 5.2, 5.3, 5.8, 5.9.

As friction decreases or equivalently $\kappa\frac{\partial^2\vec{V}}{\partial z^2}$ decreases, the angle between the wind velocity and the isobars decreases, until finally at the top of the PBL, $u \to u_g$ $v \to v_g$ and the wind blows parallel to the isobars.

We can estimate the <u>vertical</u> <u>velocity</u> at the top of the PBL.

If we assume that $\frac{\partial\rho}{\partial t} = 0$, then the continuity equation can be written

$$\nabla \cdot \rho\vec{V} = 0$$

or

$$\frac{\partial}{\partial z}\rho w = -\Big[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\Big].$$

Integrating through the Ekman layer depth gives

$$\int_0^{D_e} \frac{\partial}{\partial z}(\rho w)dz = -\int_0^{D_e}\Big[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)\Big]dz$$

or

$$\rho w(D_e) - \rho \underbrace{w(0)}_{=zero} = -\int \qquad \downarrow$$

or

$$\rho w(D_e) = -\int \qquad \downarrow$$

since $w(0) = 0$.

If we substitute the Ekman layer solutions for u and v (where $v_g = 0$, ζ is assumed constant wrt z), noting that $\nabla \cdot \vec{V}_g = 0$, and therefore

$$\frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} = 0$$

(u is independent of x at top of Ekman spiral), we get

$$\rho w(D_e) = -\frac{\partial}{\partial y}\int_0^{D_e} \rho u_g \mathrm{e}^{-\pi z/D_e}\sin\frac{\pi z}{D_e}dz \simeq -\rho\frac{\partial u_g}{\partial y}\Big(\frac{\kappa}{2f}\Big)^{1/2}$$

or

$$w(\text{at top of Ekman spiral}) = \zeta_g\Big(\frac{\kappa}{2f}\Big)^{1/2}$$

where $\zeta_g = -\frac{\partial u_g}{\partial y}$.

This vertical velocity in the "interior" is called Ekman pumping. So the greater the vorticity of the mean flow, the greater the upwelling in the atmosphere (downwelling in oceans).

This shows that the vertical velocity at the top of the PBL increases with ζ_g or equivalently

$$\zeta_g = -\frac{\partial u_g}{\partial y} = \frac{\partial}{\partial y}\Big(\frac{1}{\rho f}\frac{\partial p}{\partial y}\Big)$$

where $u_g = \frac{1}{\rho f}\frac{\partial p}{\partial y}$.

Examine the diagram.

From the diagram we see that

$$-\frac{\partial u_g}{\partial y} = \zeta_g > 0$$

(since $\frac{\partial^2 p}{\partial y^2} > 0$ for $\zeta_g > 0$, & u_g is therefore decreasing with increasing y).

At $\kappa \neq 0$, $z < D_e$, and some fixed z,

$\frac{v}{u_g}$= constant, or v= constant $\times u_g$

and for u_g changing in y direction, we have

$$\frac{\partial v}{\partial y} \neq 0,$$

and for

$$\zeta_g > 0, \qquad \frac{\partial v}{\partial y} < 0.$$

There is a horizontal (y-direction) convergence of mass therefore ($\frac{\partial v}{\partial y} < 0$), and by mass conservation there must be a positive vertical velocity in the PBL.

The mass flux out of the PBL is given by

$$\rho_0 w(\text{at } D_e) = -\frac{\partial M}{\partial y} \text{ (where } M = \textstyle\int_0^{D_e} \rho_0 v dz)$$

which is equal to the convergence of the cross-isobar mass transport in the layer.

Remember (scale analysis) geostrophic motions must be characterized by

$$\frac{\partial}{\partial z}\overline{w} = 0,$$

i.e., no divergence to geostrophic flow. So (constant wrt height) vertical velocity w must occur through the depth of the fluid.

This means convergence in bottom of Ekman layer and compensating upwelling in the "interior". Such a situation arises in presence of a cyclonic gyre in the "interior".

Anticyclonic gyres in the ocean can therefore "suck" you down.

It is now possible to do Holton 3rd, Problems 5.6, 5.7 (which uses result of 5.6), 5.8, 5.9 (which is spin down of 5.8), and 5.5 (mixed boundary layer).

Thus turbulent processes in the BL do have a dramatic effect on the synoptic-scale flow above the BL.

You can now do Problems 5.4 and 5.9.

The next question is, how are the winds maintained against the dissipative effects of BL turbulence?

As we will see in Atmospheric Dynamics II, it is the meridional (N-S) temperature gradient set up by the solar differential heating that accomplishes this task.

You can actually see something similar to this happen in a teacup of loose tea, except that the centrifugal force replaces the Coriolis force.

Stir the loose tea so that it spins rapidly — the tea then piles against the wall of your teacup; level of tea is low in the centre, so pressure is low there also.

Velocity drops to zero at the bottom and middle of the cup; therefore we get BL convergence, and by mass conservation, also get upward motion out of the BL.

Stop stirring and tea stops spinning, velocity goes to zero. This is analogous to removing thermal forcing in the atmosphere. At bottom of cup, where there is more friction, tea rapidly stops spinning; the PGF makes tea drift towards centre, and tea leaves pile up, indicating convergence.

"Tea Cup"