

# 1 The model equations

You will develop a numerical model and use it to simulate a convective flow. The model is based on the quasi-compressible outflow model (QCOM) described in Droege-meier and Wilhelmson (1987). The model predicts the horizontal velocity ( $v$ ), the vertical velocity ( $w$ ), the potential temperature ( $\theta$ ), and the non-dimensional perturbation pressure ( $\pi_1$ ). The compressible, non-rotating, adiabatic equations in Cartesian coordinates ( $y, z$ ) are:

$$\frac{\partial v}{\partial t} = -v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} - c_p \theta_0 \frac{\partial \pi_1}{\partial y} + D_v, \quad (1)$$

$$\frac{\partial w}{\partial t} = -v \frac{\partial w}{\partial y} - w \frac{\partial w}{\partial z} - c_p \theta_0 \frac{\partial \pi_1}{\partial z} + g \left( \frac{\theta}{\theta_0} - 1 \right) + D_w, \quad (2)$$

$$\frac{\partial \theta}{\partial t} = -v \frac{\partial \theta}{\partial y} - w \frac{\partial \theta}{\partial z} + D_\theta, \quad (3)$$

$$\frac{\partial \pi_1}{\partial t} = -\frac{c_s^2}{c_p \theta_0^2} \left[ \frac{\partial}{\partial y} (\theta_0 v) + \frac{\partial}{\partial z} (\theta_0 w) \right]. \quad (4)$$

See Klemp and Wilhelmson (1978) for a derivation of (4). In our version, we neglect the height variation of the density. In the equations above,  $\pi = (p/p_r)^{R/c_p}$ , where  $p_r = 1000$  mb,  $R$  is the gas constant for dry air, and  $c_p$  is the specific heat capacity at constant pressure for dry air, and  $c_s$  is the constant speed of sound. The terms  $D_v$ ,  $D_w$ , and  $D_\theta$  represent turbulent mixing. Variables with a subscript 0 refer to the basic state, which varies with height only. A subscript 1 indicates the departure from the basic state. The basic state is in hydrostatic balance:

$$\frac{d\pi_0}{dz} = -\frac{g}{c_p \theta_0}. \quad (5)$$

$$\frac{\partial v}{\partial t} = -v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} - c_p \theta_0 \frac{\partial \pi_1}{\partial y} + D_v, \quad (1)$$

$$\frac{\partial w}{\partial t} = -v \frac{\partial w}{\partial y} - w \frac{\partial w}{\partial z} - c_p \theta_0 \frac{\partial \pi_1}{\partial z} + g \left( \frac{\theta}{\theta_0} - 1 \right) + D_w, \quad (2)$$

$$\frac{\partial \theta}{\partial t} = -v \frac{\partial \theta}{\partial y} - w \frac{\partial \theta}{\partial z} + D_\theta, \quad (3)$$

$$\frac{\partial \pi_1}{\partial t} = -\frac{c_s^2}{c_p \theta_0^2} \left[ \frac{\partial}{\partial y} (\theta_0 v) + \frac{\partial}{\partial z} (\theta_0 w) \right]. \quad (4)$$

## 2 Quasi-compressibility approximation

The quasi-compressibility approximation involves artificially slowing down the sound wave modes so that a larger time step may be used. For large  $c_s$ , the solution of the equations approaches that of the anelastic system. For small  $c_s$  (near the speed of the fastest non-acoustic signal in the simulation), incorrect results occur due to artificially strong coupling between gravity wave and acoustic modes. Droegemeier and Wilhelmson found that for  $c_s > 50$  m/s, elastic energy is much less than the kinetic energy, and solutions are essentially independent of  $c_s$ .

## 4 Turbulence closure

For simplicity, we will use the eddy viscosity approach. Then terms  $D_v$ ,  $D_w$ , and  $D_\theta$  each have the form

$$K_\phi \nabla^2 \phi,$$

or, in finite-difference form,

$$K_\phi [\delta_y(\delta_y \phi) + \delta_z(\delta_z \phi)],$$

where  $K_\phi$  is the eddy diffusivity.

## 3 Finite-difference equations

The grid is staggered, with  $\pi$  and  $\theta$  located at the central point of the stencil,  $w$  one-half grid interval above and below the central point, and  $v$  one-half grid interval to the left and right of the central point. We will use the operator notation:

$$\delta_{n\xi} \phi(\xi) = \frac{\phi(\xi + n\Delta\xi/2) - \phi(\xi - n\Delta\xi/2)}{n\Delta\xi},$$

$$\overline{\phi(\xi)}^{n\xi} = \frac{\phi(\xi + n\Delta\xi/2) + \phi(\xi - n\Delta\xi/2)}{2},$$

where  $\phi$  denotes a dependent variable,  $\xi$  is the independent variable,  $\Delta\xi$  is a grid interval, and  $n\Delta\xi$  is the interval over which the operation takes place.

# Time differencing

Each of (1)-(4) can be written as

$$\frac{\partial \phi}{\partial t} = f_{\phi},$$

which can be approximated with the second-order Adams-Bashforth scheme as

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{3}{2}f_{\phi}^n - \frac{1}{2}f_{\phi}^{n-1}.$$

Here  $\Delta t$  is the time step, and the superscripts indicate the time levels. Because the Adams-Bashforth scheme is a two-level scheme, it cannot be used for the first time step of a simulation. Instead, the forward scheme may be used:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = f_{\phi}^n.$$

# Time differencing

The time step is based on the CFL criterion for linear sound waves:

$$\Delta t < \frac{(\Delta y \Delta z)^{1/2}}{c_s}.$$

For example, if  $\Delta y = \Delta z = 100$  m and  $c_s = 100$  m/s, then  $\Delta t < 1$  s for stability. Due to other aspects of the finite-difference scheme, the  $\Delta t$  that is actually used is usually smaller by a factor of  $1/2$  to  $1/4$ .

**For the diffusion equation, a different criterion applies:**

Use  $\Delta t = 0.2(\Delta z)^2 / K_\theta$ ,

where  $\Delta t$ ,  $\Delta z$ , and  $K_\theta$  are all in MKS (meter-kilogram-second) units.

# Simulation 1: Conduction

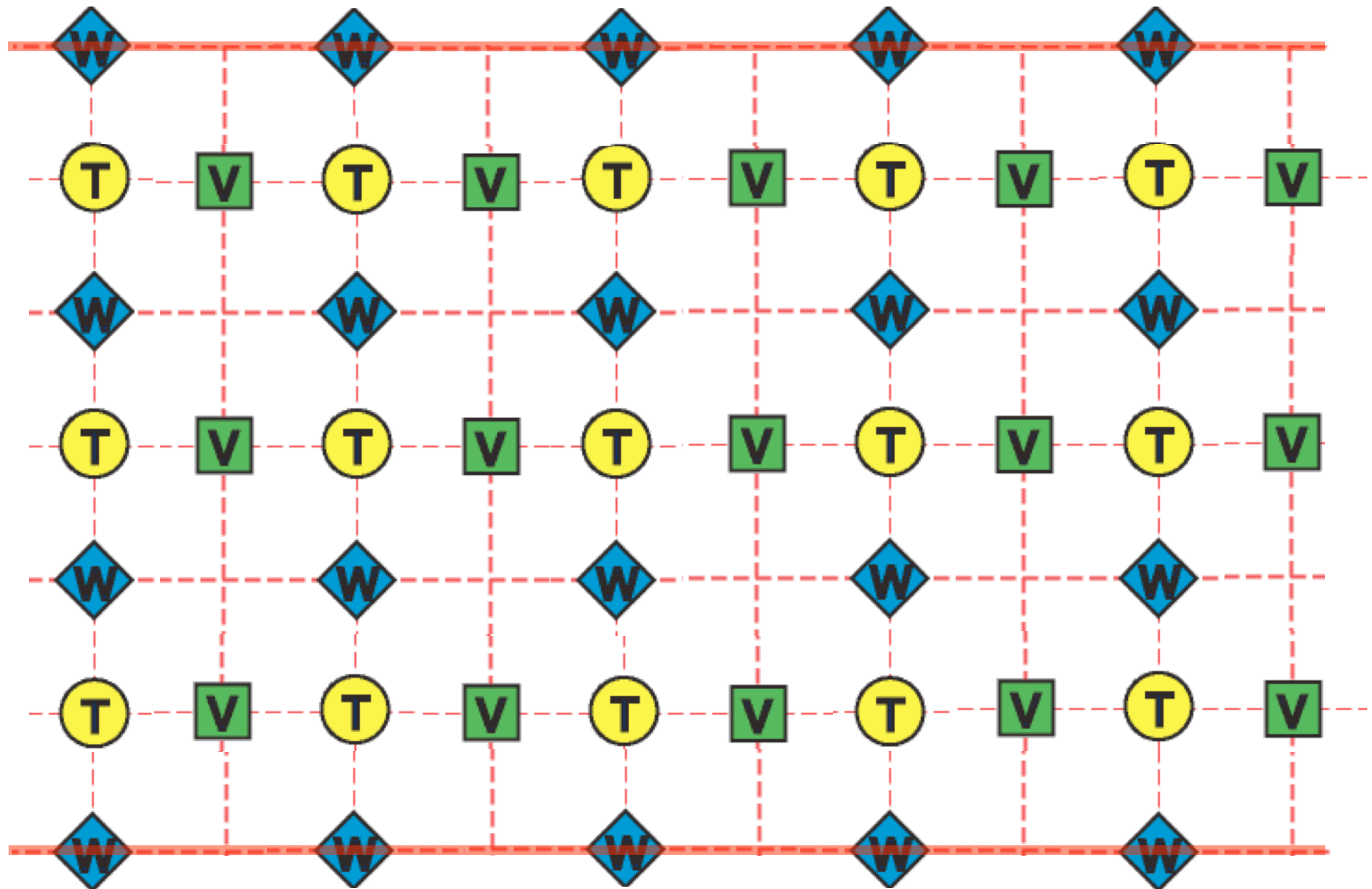
This case will correspond to the conduction regime. In this regime, there is no fluid motion, so only Eq. (3) is needed. The (potential) temperature will be specified to be steady and horizontally uniform at the upper and lower boundaries. Under these conditions, Eq. (3) simplifies to

$$\frac{\partial \theta}{\partial t} = D_\theta = K_\theta \frac{\partial^2 \theta}{\partial z^2}. \quad (10)$$

**Analytically determine the general steady-state ( $\partial \theta / \partial t = 0$ ) solution of (10) that satisfies the boundary conditions.**

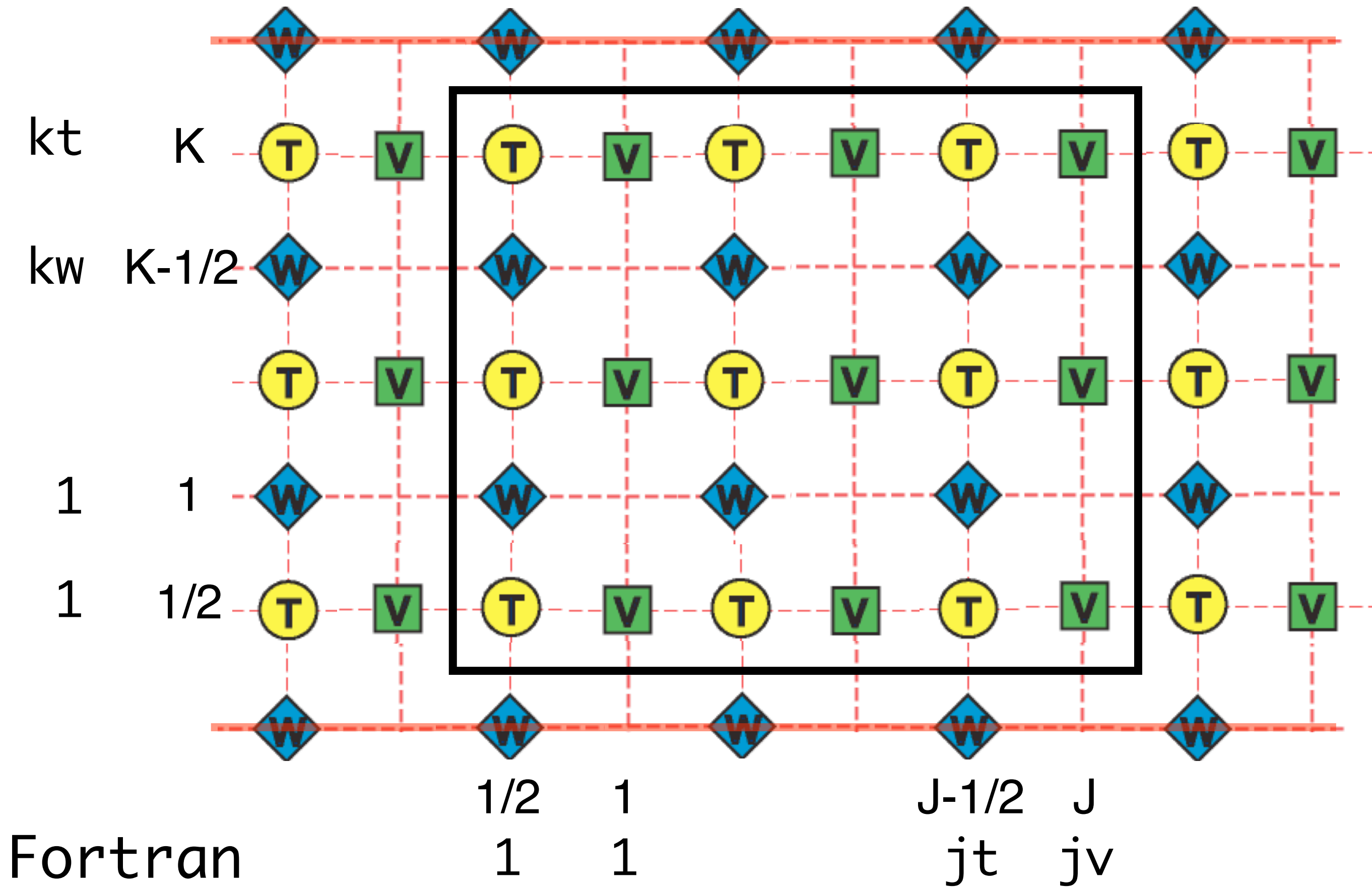
The following parameters, along with  $K_v = K_w = K_\theta$ , correspond to  $Ra=500$ , which is subcritical for free-slip boundary conditions, when  $K_v$ ,  $K_w$ , and  $K_\theta$  are interpreted as molecular diffusivities:  $H = 500$  m,  $L = H$ ,  $K_\theta = 100$  m<sup>2</sup> s<sup>-1</sup>,  $\theta_0(0) = 288$  K,  $\theta_0(H) = \theta_0(0) - \Delta\theta$ ,  $\Delta\theta = 1.2$  K. **Use your analytic, general, steady-state solution to calculate the specific solution for these parameters.**

# Staggered Grid

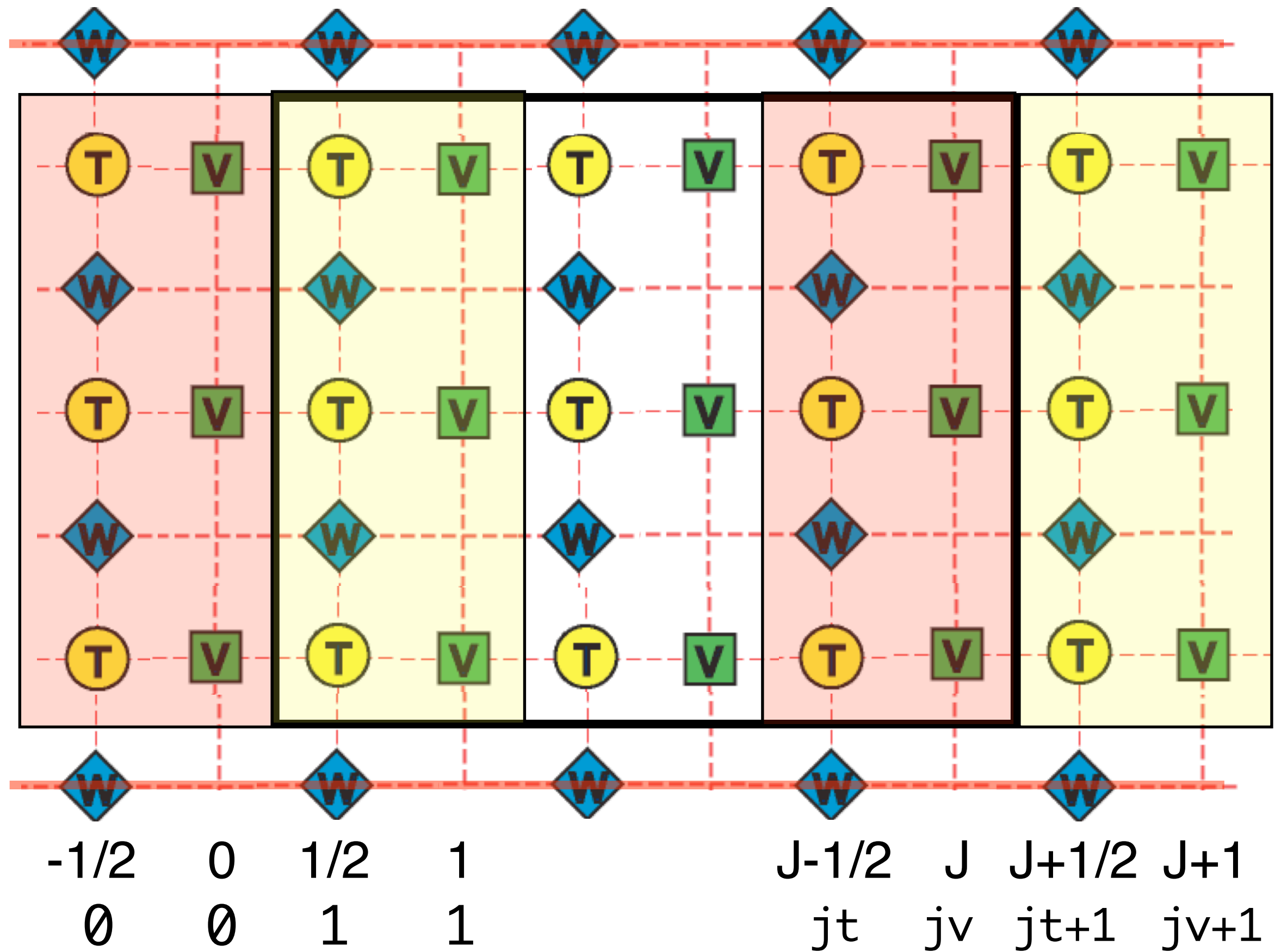




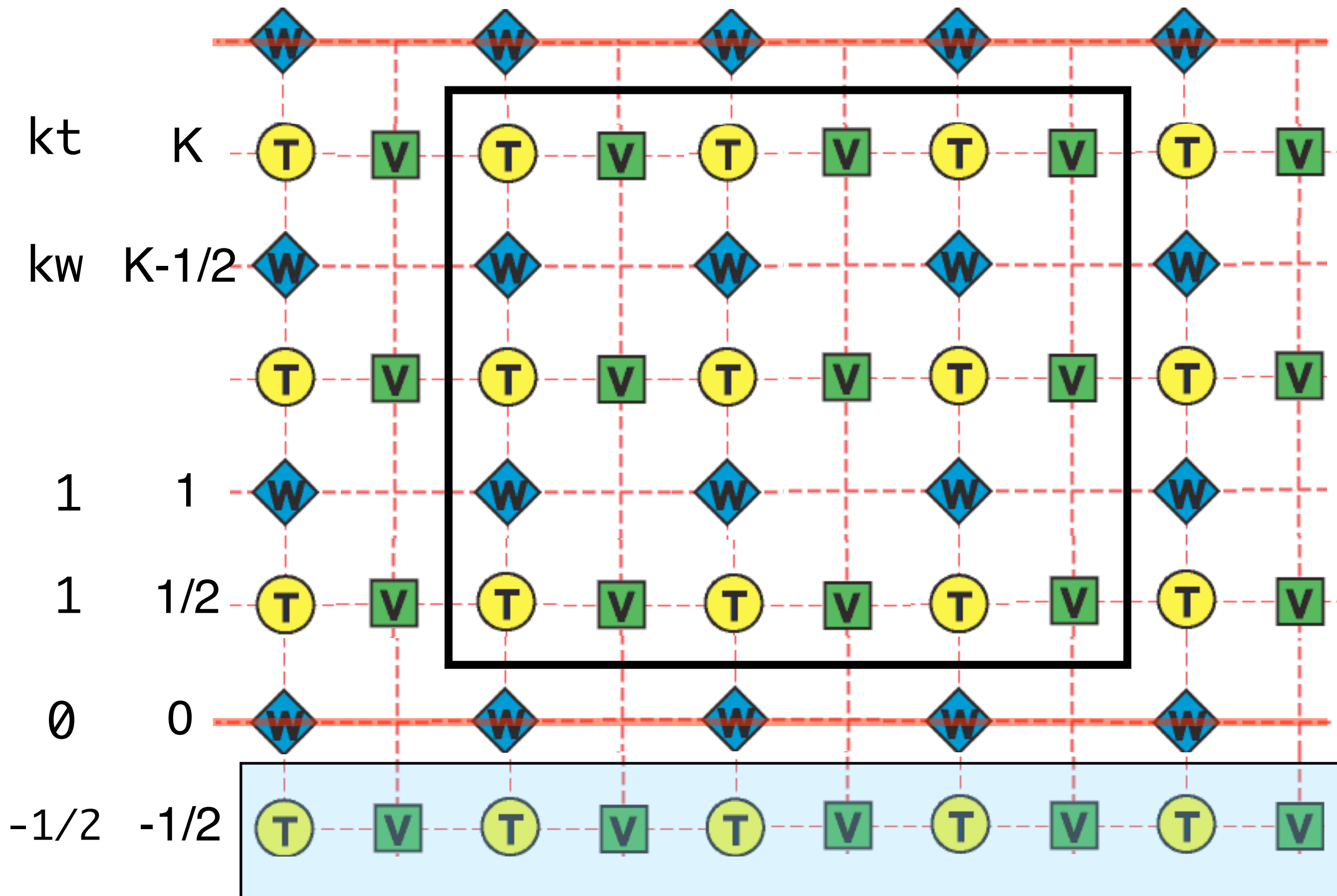
# Predicted Variable Ranges



# Cyclic Boundary Conditions



# Lower Boundary Conditions



# Upper and Lower Boundary Conditions

**Free-slip:**  $\partial v / \partial z = 0$  at  $z = 0, H$ . Thus,  $\delta_z v = 0$  at  $z = 0, H$ , and  $v_{j,k+1/2} - v_{j,k-1/2} = 0$  for  $k = 0, K$ .

**No-slip:**  $v = 0$  at  $z = 0, H$ . Thus,  $\bar{v}^z = 0$  at  $z = 0, H$ , and  $v_{j,k+1/2} + v_{j,k-1/2} = 0$  for  $k = 0, K$ .

**Non-conducting:**  $\partial \theta / \partial z = 0$  at  $z = 0, H$ . Thus,  $\delta_z \theta = 0$  at  $z = 0, H$ , and  $\theta_{j,k+1/2} - \theta_{j,k-1/2} = 0$  for  $k = 0, K$ .

**Conducting:**  $\theta = \theta_0$  at  $z = 0, H$ . Thus,  $\bar{\theta}^z = \theta_0$  at  $z = 0, H$ , and  $(\theta_{j,k+1/2} + \theta_{j,k-1/2})/2 = (\theta_0)_k$  for  $k = 0, K$ .

# Lower Boundary Conditions

