A Simple Turbulence Closure Model

Atmospheric Sciences 6150

1 Cartesian Tensor Notation

Reynolds decomposition of velocity:

$$\mathbf{V} = \overline{\mathbf{V}} + \mathbf{v} \Rightarrow \mathbf{V} = U_i + u_i$$

Mean velocity:

$$\overline{\mathbf{V}} = U\mathbf{i} + V\mathbf{j} + W\mathbf{k} = (U, V, W) \Rightarrow U_i = (U_1, U_2, U_3)$$

Turbulent velocity:

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = (u, v, w) \Rightarrow u_i = (u_1, u_2, u_3)$$

Gradient operator:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Rightarrow \frac{\partial}{\partial x_k} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$$

Advection operator:

$$\overline{\mathbf{V}} \cdot \nabla = U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + W\frac{\partial}{\partial z} \Rightarrow U_k\frac{\partial}{\partial x_k} = U_1\frac{\partial}{\partial x_1} + U_2\frac{\partial}{\partial x_2} + U_3\frac{\partial}{\partial x_3}$$

The covariance matrix is a tensor of rank 2:

$$\begin{pmatrix} \overline{uu} & \overline{uv} & \overline{uw} \\ \overline{vu} & \overline{vv} & \overline{vw} \\ \overline{wu} & \overline{wv} & \overline{ww} \end{pmatrix} \Rightarrow \overline{u_i u_j} = \begin{pmatrix} \overline{u_1 u_1} & \overline{u_1 u_2} & \overline{u_1 u_3} \\ \overline{u_2 u_1} & \overline{u_2 u_2} & \overline{u_2 u_3} \\ \overline{u_3 u_1} & \overline{u_3 u_2} & \overline{u_3 u_3} \end{pmatrix}$$

Turbulent kinetic energy, $e = q^2/2$, and summation over repeated indices:

$$q^2 = \overline{uu} + \overline{vv} + \overline{ww} \Rightarrow q^2 = \overline{u_i u_i} = \overline{u_1 u_1} + \overline{u_2 u_2} + \overline{u_3 u_3}$$

A tensor of rank 3:

$$\left(\begin{pmatrix} \overline{uuu} & \overline{uuv} & \overline{uuw} \\ \overline{uvu} & \overline{uvv} & \overline{uvw} \\ \overline{uwu} & \overline{uwv} & \overline{uww} \end{pmatrix}, \begin{pmatrix} \overline{vuu} & \overline{vuv} & \overline{vuw} \\ \overline{vvu} & \overline{vvv} & \overline{vvw} \\ \overline{vwu} & \overline{vwv} & \overline{vww} \end{pmatrix}, \begin{pmatrix} \overline{wuu} & \overline{wuv} & \overline{wuw} \\ \overline{wvu} & \overline{wvv} & \overline{wvw} \\ \overline{wwu} & \overline{wwv} & \overline{www} \end{pmatrix}\right)$$

$$\Rightarrow \overline{u_i u_j u_k} = (\overline{u_1 u_j u_k}, \overline{u_2 u_j u_k}, \overline{u_3 u_j u_k})$$

$$= \left(\begin{pmatrix} \overline{u_1u_1u_1} & \overline{u_1u_1u_2} & \overline{u_1u_1u_3} \\ \overline{u_1u_2u_1} & \overline{u_1u_2u_2} & \overline{u_1u_2u_3} \\ \overline{u_1u_3u_1} & \overline{u_1u_3u_2} & \overline{u_1u_3u_3} \end{pmatrix}, \begin{pmatrix} \overline{u_2u_1u_1} & \overline{u_2u_1u_2} & \overline{u_2u_1u_3} \\ \overline{u_2u_2u_1} & \overline{u_2u_2u_2} & \overline{u_2u_2u_3} \\ \overline{u_2u_3u_1} & \overline{u_2u_3u_2} & \overline{u_2u_3u_3} \end{pmatrix}, \begin{pmatrix} \overline{u_3u_1u_1} & \overline{u_3u_1u_2} & \overline{u_3u_1u_3} \\ \overline{u_3u_2u_1} & \overline{u_3u_2u_2} & \overline{u_3u_2u_3} \\ \overline{u_3u_3u_1} & \overline{u_3u_3u_2} & \overline{u_3u_3u_3} \end{pmatrix}\right)$$

Kronecker delta:

$$\delta_{ij} = \left\{ \begin{array}{ll} 1 & i = j \\ 0 & i \neq j \end{array} \right\}$$

First moments of velocity (3 unique):

$$U_i$$

Second moments of velocity (9, 6 are unique):

$$\overline{u_i u_j}$$

Third moments of velocity (27, 10 are unique):

$$\overline{u_i u_j u_k}$$

2 The Closure Problem

The momentum equation for a homogeneous incompressible fluid at high Re is

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla \left(\frac{\mathcal{P}}{\rho_0} \right) + \nu \nabla^2 \mathbf{V}. \tag{1}$$

Decompose the velocity and pressure into means and deviations:

$$\mathbf{V} = U_i + u_i$$

$$\mathcal{P} = P + p$$

Substitute into (1) and average:

$$\frac{\partial U_j}{\partial t} + U_k \frac{\partial U_j}{\partial x_k} = -\frac{\partial}{\partial x_j} \left(\frac{P}{\rho_0} \right) - \frac{\partial \overline{u_k u_j}}{\partial x_k}. \tag{2}$$

The additional term is due to momentum transport by the turbulent velocity fluctuations.

One way to close the equations is to assume that

$$\overline{u_k u_j} = -K_m \left(\frac{\partial U_k}{\partial x_j} + \frac{\partial U_j}{\partial x_k} \right).$$

This is the *eddy viscosity model*. However, K_m is a property of the flow, not of the fluid (as viscosity is), and is not necessarily a constant (as viscosity is). Using a constant K_m is generally not a good assumption in the atmosphere.

Another way to close the equations is to derive equations for the Reynolds stresses, $\overline{u_k u_j}$, and to make assumptions for the unknown terms in these equations. This is a *second-moment (or second-order) closure model.*

3 Closure Models

First-order closure model :

$$\left\{\begin{array}{l}\overline{u_iu_j}\\ \overline{u_i\theta}\end{array}\right\} \Rightarrow \text{diagnostically related to} \left\{\begin{array}{l}U_i\\ \theta\end{array}\right\} \Rightarrow \text{prognostically determined}$$

Second-order closure model :

$$\left\{\begin{array}{l}\overline{u_iu_ju_k}\\ \overline{p(u_{i,j}+u_{j,i})}\\ \text{etc.}\end{array}\right\} \Rightarrow \text{diagnostically related to} \left\{\begin{array}{l}\overline{u_iu_j}\\ \overline{u_i\theta}\\ \overline{\theta^2}\end{array}\right\} \Rightarrow \text{prognostically determined}$$

Third-order closure model :

$$\left\{\begin{array}{l}\overline{u_iu_ju_ku_l}\\ \overline{p((u_iu_j)_{,k}+\ldots)}\\ \text{etc.}\end{array}\right\} \Rightarrow \text{diagnostically related to} \left\{\begin{array}{l}\overline{u_iu_j}\\ \overline{u_i\theta}\\ \overline{\theta^2}\\ \overline{u_iu_ju_k}\\ \overline{u_iu_j\theta}\\ \overline{u_i\theta^2}\\ \overline{\theta^3}\end{array}\right\} \Rightarrow \text{prognostically determined}$$

4 A Simple Turbulence Closure Model

We use the eddy viscosity model for the turbulent fluxes. We set the eddy viscosity K_m to be proportional to the turbulence velocity scale q times a turbulence length scale l. This allows K_m to depend on the turbulence properties, which is a more realistic than using a constant K_m.

Unknown turbulent fluxes	**Modeling assumption**
$\overline{u_i u_j}$ (momentum)	$\frac{q^2}{3}\delta_{ij} - q l_1 \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$
$\overline{u_i \theta}$ (any scalar)	$-q l_2 \frac{\partial \Theta}{\partial x_i}$

To close these models for the turbulent fluxes, we require an equation for $q^2 \equiv \overline{u_i u_i}$. To do this, we start with the full equation for q^2:

$$\frac{dq^2}{dt} = -\frac{\partial \overline{u_i u_i u_j}}{\partial x_j} - \frac{2}{\rho}\frac{\partial \overline{p u_j}}{\partial x_j} - 2\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} + 2\frac{g_i}{\Theta}\overline{u_i \theta_v} - 2\epsilon.$$

The terms on the r.h.s. of this equation represent turbulent transport, pressure transport, shear (mechanical) production, buoyancy production (or loss), and dissipation, respectively. To close this equation we

1. Assume that production and dissipation balance:

$$0 = SP + BP - D.$$

2. Use the models above for the fluxes.

3. Model dissipation using

$$\epsilon = \frac{q^3}{\Lambda_1}.$$

The result is

$$q^2 = \Lambda_1 l_1 \left[S_{ij}\frac{\partial U_i}{\partial x_j} - \frac{l_2}{l_1}\frac{g_i}{\Theta}\frac{\partial \Theta_v}{\partial x_i}, \right] \tag{3}$$

where

$$S_{ij} \equiv \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$$

and

$$g_i = (0, 0, g).$$

Let $l_1 = A_1 l$, $l_2 = A_2 l$, and $\Lambda_1 = B_1 l$, where l is the turbulence length scale. $A_1 = 0.92$, $A_2 = 0.74$, and $B_1 = 16.6$ are constants determined from experiments. Many prescriptions for the turbulent length scale l exist. The only definite constraint is that $l \to kz$ near the surface so that $K_m = ku_* z$ under neutral conditions. One commonly used form is

$$l = \frac{l_\infty}{1 + l_\infty / kz},$$

where the asymptotic length scale l_∞ is specified to be about 10 percent of the boundary layer depth. The specification of l is usually not very critical.

5 Richardson Number Dependence

Equation (3) includes the effects of stratification, so it should exhibit a dependence on Richardson number. To show this, we will first simplify (3) by making the boundary layer approximation:

$$U_3 = 0, \ \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} = 0.$$

Then only

$$S_{13} = S_{31} = \frac{\partial U_1}{\partial x_3}$$

and

$$S_{23} = S_{32} = \frac{\partial U_2}{\partial x_3}$$

are nonzero, so

$$S_{ij} \frac{\partial U_i}{\partial x_j} = \left(\frac{\partial U_1}{\partial x_3} \right)^2 + \left(\frac{\partial U_2}{\partial x_3} \right)^2.$$

Also,

$$\frac{g_i}{\Theta} \frac{\partial \Theta_v}{\partial x_i} = \frac{g}{\Theta} \frac{\partial \Theta_v}{\partial x_3}.$$

Using these simplifications in (3), we obtain

$$q^2 = \Lambda_1 l_1 \left[\left(\frac{\partial U_1}{\partial x_3} \right)^2 + \left(\frac{\partial U_2}{\partial x_3} \right)^2 - \frac{l_2}{l_1} \frac{g}{\Theta} \frac{\partial \Theta_v}{\partial x_3} \right]. \tag{4}$$

The condition for $q^2 > 0$ is therefore

$$\left(\frac{\partial U_1}{\partial x_3}\right)^2 + \left(\frac{\partial U_2}{\partial x_3}\right)^2 - \frac{l_2}{l_1}\frac{g}{\Theta}\frac{\partial \Theta_v}{\partial x_3} > 0.$$

Write this in terms of a *gradient Richardson number* Ri:

$$\frac{l_1}{l_2} > \frac{\frac{g}{\Theta}\frac{\partial \Theta_v}{\partial x_3}}{\left(\frac{\partial U_1}{\partial x_3}\right)^2 + \left(\frac{\partial U_2}{\partial x_3}\right)^2} \equiv \mathrm{Ri}$$

or

$$\mathrm{Ri} < \frac{l_1}{l_2} = \frac{A_1 l}{A_2 l} = \frac{A_1}{A_2} = \frac{0.92}{0.74} = 1.24.$$

Theoretical and laboratory results suggest that laminar flow becomes turbulent when

$$\mathrm{Ri} < 0.25,$$

and that turbulent flow becomes laminar when

$$\mathrm{Ri} > 1.$$

These are *local* criteria. Even if $\mathrm{Ri} > 1$ when estimated using resolved variables, it may be < 1 locally within a grid volume.

6 Performance

This simple closure model works best when its assumption of a local balance between production and dissipation is most nearly met. This condition is most likely to be valid when and where shear production dominates buoyancy production.