Table 4.1 Molecular thermal properties of natural materials[a].

Material	Condition	Mass density ρ ($kg\ m^{-3} \times 10^3$)	Specific heat c ($J\ kg^{-1}\ K^{-1} \times 10^3$)	Heat capacity C ($J\ m^{-3}\ K^{-1} \times 10^6$)	Thermal conductivity k ($W\ m^{-1}\ K^{-1}$)	Thermal diffusivity α_h ($m^2\ s^{-1} \times 10^{-6}$)
Air	20°C, Still	0.0012	1.01	0.0012	0.025	20.5
Water	20°C, Still	1.00	4.18	4.18	0.57	0.14
Ice	0°C, Pure	0.92	2.10	1.93	2.24	1.16
Snow	Fresh	0.10	2.09	0.21	0.08	0.38
Snow	Old	0.48	2.09	0.84	0.42	0.05
Sandy soil	Fresh	1.60	0.80	1.28	0.30	0.24
(40% pore space)	Saturated	2.00	1.48	2.96	2.20	0.74
Clay soil	Dry	1.60	0.89	1.42	0.25	0.18
(40% pore space)	Saturated	2.00	1.55	3.10	1.58	0.51
Peat soil	Dry	0.30	1.92	0.58	0.06	0.10
(80% pore space)	Saturated	1.10	3.65	4.02	0.50	0.12
Rock	Solid	2.70	0.75	2.02	2.90	1.43

[a]After Oke (1987) and Garratt (1992).

4.4 Theory of Soil Heat Transfer

Here we consider a uniform conducting medium (soil), with heat flowing only in the vertical direction. Let us consider the energy budget of an elemental volume consisting of a cylinder of horizontal cross-section area ΔA and depth Δz, bounded between the levels z and $z + \Delta z$, as shown in Figure 4.3.

Heat flow in the volume at depth $z = H\Delta A$.

Heat flow out of the volume at $z + \Delta z = [H + (\partial H/\partial z)\,\Delta z]\Delta A$.

The net rate of heat flow in the control volume $= -(\partial H/\partial z)\Delta z\,\Delta A$.

The rate of change of internal energy within the control volume $= (\partial/\partial t)(\Delta A \Delta z C_s T)$, where C_s is the volumetric heat capacity of soil.

According to the law of conservation of energy, if there are no sources or sinks of energy within the elemental volume, the net rate of heat flowing in the volume should equal the rate of change of internal energy in the volume, so that

$$(\partial/\partial t)(C_s T) = -\partial H/\partial z \tag{4.2}$$

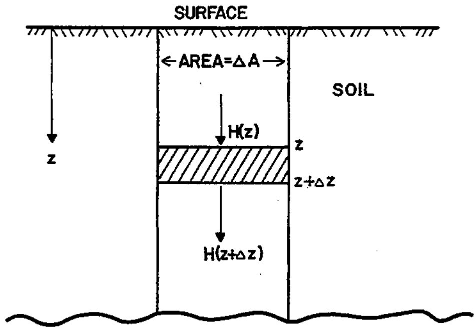

Figure 4.3 Schematic of heat transfer in a vertical column of soil below a flat, horizontal surface.

Further, assuming that the heat capacity of the medium does not vary with time, and substituting from Equation (4.1) into (4.2), we obtain Fourier's equation of heat conduction:

$$\partial T/\partial t = (\partial/\partial z)\,[(k/C_s)(\partial T/\partial z)] = (\partial/\partial z)[\alpha_h(\partial T/\partial z)] \tag{4.3}$$

The one-dimensional heat conduction equation derived here can easily be generalized to three dimensions by considering the net rate of heat flow in an elementary control volume $\Delta x \Delta y \Delta z$ from all the directions. In our applications involving heat transfer through soils, however, we will be primarily concerned with the one-dimensional Equation (4.2) or (4.3).

Equation (4.2) can be used to determine the ground heat flux H_G in the energy balance equation from measurements of soil temperatures as functions of time, at various depths below the surface. The method is based on the integration of Equation (4.2) from $z = 0$ to D

$$H_G = H_D + \int_0^D \frac{\partial}{\partial t}(C_s T)\, dz \tag{4.4}$$

where D is some reference depth where the soil heat flux H_D is either zero (e.g., if at $z = D$, $\partial T/\partial z = 0$) or can be easily estimated [e.g., using Equation (4.1)]. The former is preferable whenever feasible, because it does not require a knowledge of thermal conductivity, which is more difficult to measure than heat capacity.

4.5 Thermal Wave Propagation in Soils

The solution of Equation (4.3), with given initial and boundary conditions, is used to study theoretically the propagation of thermal waves in soils and other substrata. For any arbitrary prescription of surface temperature as a function of time and soil layers, the solution to Equation (4.3) can be obtained numerically. Much about the physics of thermal wave propagation can be learned, however, from a simple analytic solution which is obtained when the surface temperature is specified as a sinusoidal function of time and the subsurface medium is assumed to be homogeneous throughout the depth of wave propagation

$$T_s = T_m + A_s \sin[(2\pi/P)(t - t_m)] \tag{4.5}$$

Here, T_m is the mean temperature of the surface or submedium, A_s and P are the amplitude and period of the surface temperature wave, and t_m is the time when $T_s = T_m$, as the surface temperature is rising.

The solution of Equation (4.3) satisfying the boundary conditions that at $z = 0$, $T = T_s(t)$, and as $z \to \infty$, $T \to T_m$, is given by

$$T = T_m + A_s \exp(-z/d) \sin[(2\pi/P)(t - t_m) - z/d] \tag{4.6}$$

which the reader may verify by substituting in Equation (4.3). Here, d is the damping depth of the thermal wave, defined as

$$d = (P\alpha_h/\pi)^{1/2} \tag{4.7}$$

Note that the period of thermal wave in the soil remains unchanged, while its amplitude decreases exponentially with depth ($A = A_s \exp(-z/d)$); at $z = d$ the wave amplitude is reduced to about 37% of its value at the surface and at $z = 3d$ the amplitude decreases to about 5% of the surface value. The phase lag relative to the surface wave increases in proportion to depth (phase lag $= z/d$), so that there is a complete reversal of the wave phase at $z = \pi d$. The corresponding lag in the time of maximum or minimum in temperature is also proportional to depth (time lag $= zP/2\pi d$).

The results of the above simple theory would be applicable to the propagation of both the diurnal and annual temperature waves through a homogeneous submedium, provided that the thermal diffusivity of the medium remains constant over the whole period, and the surface temperature wave is nearly sinusoidal. For the diurnal period, the latter condition is usually not satisfied especially during the nighttime period when the wave is observed to be more asymmetric around its minimum value (Rosenberg *et al.*, 1983, Chapter 2). Also, disturbed weather conditions with clouds and precipitation are likely to alter the surface temperature wave, as well as thermal diffusivity due to changes in the soil moisture content. For the annual period, the assumption of constant thermal diffusivity may be even more questionable, except for bare soils in arid regions. Note that the damping depth, which is a measure of the extent of thermal wave propagation, for the annual wave is expected to be $\sqrt{365} = 19.1$ times the damping depth for the diurnal wave.

Example Problem 1

Calculate the damping depth and the depth of thermally active soil layer where there is a complete reversal of the phase of the diurnal thermal wave from that at the surface for the following types of soils:
(a) dry sandy soil; (b) saturated sandy soil (40% pore space);
(c) dry clay soil; (d) dry peat soil.
What will be the amplitude of the thermal wave at the depth of the phase reversal relative to that at the surface?

Solution

Using the thermal diffusivities given in Table 4.1, Equation (4.7) for damping depth d, and the depth of phase reversal as πd, one can obtain the following results:

(a) For dry sandy soil, $\alpha_h = 0.24 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$.
Damping depth $d = (24 \times 3600 \times 0.24 \times 10^{-6}/\pi)^{1/2} = 0.081$ m.
Depth of phase reversal $= \pi d = 0.25$ m.
(b) For saturated sandy soil, $\alpha_h = 0.74 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$.
Damping depth $d = 0.143$ m.
Depth of phase reversal $= \pi d = 0.45$ m.
(c) For dry clay soil, $\alpha_h = 0.18 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$.
Damping depth $d = 0.070$ m.
Depth of phase reversal $= \pi d = 0.22$ m.
(d) For dry peat soil, $\alpha_h = 0.10 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$.
Damping depth $d = 0.052$ m.
Depth of phase reversal $= \pi d = 0.16$ m.

In all the cases the amplitude of the thermal wave at $z = \pi d$, relative to that at the surface, is

$$A/A_s = \exp(-\pi\, d/d) = \exp(-\pi) \cong 0.042$$

One can conclude that dry peat soil offers the maximum resistance (minimum d) and saturated sandy soil the minimum resistance (maximum d) to the propagation of the diurnal thermal wave.

The observed temperature waves (Figures 4.1 and 4.2) in bare, dry soils are found to conform well to the pattern predicted by the theory. In particular, the observed annual waves show nearly perfect sinusoidal forms. The plots of wave amplitude (on log scale) and phase or time lag as functions of depth (both on linear scale) are well represented by straight lines (Figure 4.4) whose slopes determine the damping depth and, hence, thermal diffusivity of the soil.

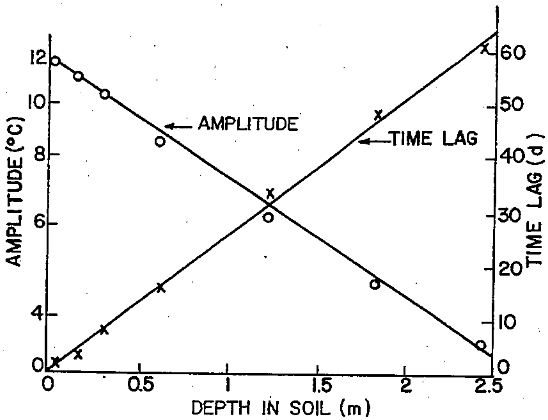

Figure 4.4 Variations of amplitude and time lag of the annual soil temperature waves with depth in the soil. [From Deacon (1969).]

Example Problem 2

Using the weekly averaged data on soil temperatures at different depths obtained by West (1952), the amplitude and time lag of thermal waves are plotted as functions of depth in soil in Figure 4.4. Estimate the damping depth and thermal diffusivity of the soil from the best-fitted lines through the data points.

Solution

Note that according to Equation (4.6), the amplitude of thermal wave decreases exponentially with depth, i.e.,

$$A = A_s \exp(-z/d)$$

or

$$\ln A = \ln A_s - \frac{z}{d}$$

Therefore, a plot of ln A (A on a log scale) against z should result in a straight line with a slope of $-1/d$. The slope of the best-fitted straight line through the amplitude data in Figure 4.4 can be estimated, from which $d \cong 2.05$ m.

Thermal diffusivity $\alpha_h = \pi d^2/P \cong 0.42 \times 10^{-6}\ \mathrm{m^2\ s^{-1}}$.

Also, according to the thermal wave equation (Equation 4.6),

$$\text{Time lag} = Pz/2\pi d$$

Therefore, a plot of time lag versus z should also result in a straight line with a slope of $P/2\pi d$. The slope of the best-fitted line throughout the time lag data points in Figure 4.4 can be estimated as 25.6 days m^{-1}, from which

$$d = P/(2\pi \times \text{slope}) \cong 2.27 \text{ m}$$

$$\alpha_h = \pi d^2/P \cong 0.51 \times 10^{-6} \text{ m}^2 \text{ s}^{-1}$$

The two estimates of d, based on amplitude and time lag data, are in fairly good agreement. The agreement between the estimated damping depths based on the top 0.30 m of the soil temperature data for the diurnal period is found to be much better (Deacon, 1969).