Note the diverse scales of eddy motion and self-similar appearance at different lengthscales of the turbulence in this water jet. Only eddies of size $0.01L$ or smaller are subject to substantial viscous dissipation.
Description of Turbulence

Turbulence is characterized by disordered, eddying fluid motions over a wide range of length-scales. While turbulent flows still obey the deterministic equations of fluid motion, a small initial perturbation to a turbulent flow rapidly grows to affect the entire flow (loss of predictability), even if the external boundary conditions such as pressure gradients or surface fluxes are unchanged. We can imagine an infinite family or ensemble of turbulent flows all forced by the same boundary conditions, but starting from a random set of initial flows. One way to create such an ensemble is by adding random small perturbations to the same initial flow, then looking at the resulting flows at a much later time when they have become decorrelated with each other.

Turbulent flows are best characterized statistically through ensemble averaging, i.e. averaging some quantity of interest across the entire ensemble of flows. By definition, we cannot actually measure an ensemble average, but turbulent flows vary randomly in time and (along directions of symmetry) in space, so a sufficiently long time or space average is usually a good approximation to the ensemble average. Any quantity \( a \) (which may depend on location or time) can be partitioned

\[
a = \bar{a} + a',
\]

where \( \bar{a} \) is the ensemble mean of \( a \), and \( a' \) is the fluctuating part or perturbation of \( a \). The ensemble mean of \( a' \) is zero by definition; \( a' \) can be characterized by a probability distribution whose spread is characterized by the variance \( a'^2 \). One commonly referred to measure of this type is the turbulent kinetic energy (TKE) per unit mass,

\[
\text{TKE} = \frac{1}{2}(u'^2 + v'^2 + w'^2)
\]

This is proportional to the variance of the magnitude of the velocity perturbation:

\[
\text{TKE} = \frac{1}{2}q', \quad q' = (u^2 + v^2 + w^2)^{1/2}.
\]

We may also be interested in covariances between two quantities \( a \) and \( b \). These might be the same field measured at different locations or times (i.e., the spatial or temporal autocorrelation), or different fields measured at the same place and time (e.g. the upward eddy heat flux is proportional to the covariance \( w'T' \) between vertical velocity \( w \) and temperature \( T \)). Variances and covariances are called second-order moments of the turbulent flow. These take a longer set of measurements to determine reliably than ensemble means.

The temporal autocorrelation of a perturbation quantity \( a' \) measured at a fixed position,

\[
R(T) = \frac{\bar{a'}(t)a'(t+T)}{\bar{a'}(t)a'(t)}
\]

can be used to define an integral time scale

\[
\tau_a = \int_0^\infty R(T)dT
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which characterizes the timescale over which perturbations of \( a \) are correlated. One may similarly
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define an integral length scale.

One commonly referred-to statistic for turbulence in which buoyancy forces are important involves third-order moments. The vertical velocity skewness is defined

\[ S = \frac{w'w'w'}{w'w^3/2} \]

The skewness is positive where perturbation updrafts tend to be more intense and narrower than perturbation downdrafts, e.g. in cumulus convection, and is negative where the downdrafts are more intense and narrower, e.g. at the top of a stratocumulus cloud. Skewness larger than 1 indicates quite noticeable asymmetry between perturbation up and downdrafts.

Fourier spectra in space or time of perturbations are commonly used to help characterize the distribution of the fluctuations over different length and time scales. For example, given a long time series of a quantity \( a(t) \), we can take the Fourier transform of its autocovariance to get its temporal power spectrum vs. frequency \( \omega \).

\[ \tilde{S}_a(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a'(t)a'(t+T)\exp(-i\omega T)\,dT}{a'(t)a'(t)} \]  

(this is real and positive for all \( \omega \))

Given the power spectrum, one can recover the autocovariance by an inverse Fourier transform, and in particular, the variance is the integral of the power spectrum over all frequencies.

\[ \int_{-\infty}^{\infty} \tilde{S}_a(\omega)\,d\omega \]

so we can think of the power spectrum as a partitioning of the variance of \( a \) between frequencies.

For spatially homogeneous turbulence one can do a 3D Fourier transform of the spatial autocovariance function to obtain the spatial power spectrum vs. wavevector \( \mathbf{k} \),

\[ \hat{S}_a(\mathbf{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} a'(\mathbf{r})a'(\mathbf{r}+\mathbf{R})\exp(-i(\mathbf{k} \cdot \mathbf{R}))\,d\mathbf{k} \]

again the variance \( a \) is the integral of the power spectrum over all wavenumbers.

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\[ \langle a'(\mathbf{r})a'(\mathbf{r}) \rangle = \int_{-\infty}^{\infty} \hat{S}_a(\mathbf{k}) d\mathbf{k} \]
If the turbulence is also **isotropic**, i.e. looks the same from all orientations, then the power spectrum depends only on the magnitude \( k \) of the wavenumber, and we can partition the variance into different wavenumber bands:

\[
as'(\mathbf{r})a'(\mathbf{r}) = \int_0^\infty \hat{S}_a(k)4\pi k^2 \, dk.
\]

In particular, for homogeneous isotropic turbulence we can partition TKE into contributions from all wavenumbers; this is called the **energy spectrum** \( E(k) \).

\[
\text{TKE} = \frac{1}{2} \hat{q}(\mathbf{r})\hat{q}(\mathbf{r}) = \int_0^\infty E(k) \, dk
\]

Roughly speaking, the energy spectrum at a particular wavenumber \( k \) can be visualized as being due to eddies whose characteristic size (diameter) is \( 2\pi/k \).

**Turbulent Energy Cascade**

Ultimately, boundary layer turbulence is due to continuous forcing of the mean flow toward a state in which shear or convective instabilities grow. These instabilities typically feed energy mostly into eddies whose characteristic size is comparable to the boundary layer depth. When these eddies become turbulent, considerable variability is also seen on much smaller scales. This is often described as an **energy cascade** from larger to smaller scales through the interaction of eddies. It is called a cascade because eddies are deformed and folded most efficiently by other eddies of comparable scales, and this squeezing and stretching transfers energy between nearby lengthscales. Thus the large eddies feed energy into smaller ones, and so on until the eddies become so small as to be viscously dissipated. There is typically a range of eddy scales larger than this at which buoyancy or shear of the mean flow are insignificant to the eddy statistics compared to the effects of other turbulent eddies; in this **inertial subrange** of scales the turbulent motions are roughly homogeneous, isotropic, and inviscid, and if fact from a photograph one could not tell at what lengthscale one is looking, i.e. the turbulence is self-similar.

Dimensional arguments have always played a central role in our understanding of turbulence due to the complexity and self-similarity of turbulent flow. Kolmogorov (1941) postulated that for large Reynolds number, the statistical properties of turbulence above the viscous dissipation scale are independent of viscosity and depend only on the rate at which energy produced at the largest scale \( L \) is cascaded down to smaller eddies and ultimately dissipated by viscosity. This is measured
turbulence is generated in the shoulder of the expansion and this leads to an abrupt loss in mechanical energy. The momentum equation, coupled to a few judicious assumptions about the nature of the flow, allows us to predict the loss of mechanical energy per unit mass. It is

\[
\text{(loss of energy)} = \frac{1}{2} (V_1 - V_2)^2
\]

where \(V_1\) and \(V_2\) are the mean velocities in the pipe at sections 1 and 2. This is the famous 'Borda head loss' equation. (Try deriving this for yourself—see Exercise 3.6.) Now the important thing about the expression above is that, just like equation (3.6), the rate of loss of energy seems to be independent of \(\nu\).

Some instructors like to tease undergraduates about this. How can the dissipation be independent of \(\nu\)? It is always reassuring to see how quickly those who have already met the boundary layer see the point. Recall that the key feature of a laminar boundary layer is that, no matter how small \(\nu\) might be, the viscous shear stresses are always comparable with the inertial forces. This has to be so since it is the viscous stresses which pull the velocity down to zero at the surface, allowing the no-slip condition to be met. Thus, if we make \(\nu\) smaller and smaller, nature simply retaliates by making the boundary layer thinner and thinner, and it does it in such a way that \(\nu \partial u_i / \partial y\) remains an order one quantity: \(\nu \partial u_i / \partial y \sim (\nu \mathbf{\nabla}) \text{a} \). Thus, in a laminar boundary layer, \(\text{Re} = \omega \delta / \nu\) is always of the order of unity (Figure 3.8(a)). The same sort of thing is happening in the pipe expansion and in grid turbulence. As we let \(\nu\) become progressively smaller we might expect that the dissipation becomes less, but it does not. Instead we find that finer and finer structures appear in the fluid, and these fine-scale structures have a thickness which is just sufficient to ensure that the rate of viscous dissipation per unit mass remains finite and equal to \(\sim \nu^2 / l\). In short, the vorticity distribution becomes increasingly singular (and intermittent) as \(\text{Re} \to \infty\), consisting of extremely thin sheets and tubes of intense vorticity.

We might also interpret (3.6) in terms of Richardson's energy cascade, which was introduced in Chapter 1, Section 1.6. Recall that most of the energy is held in large-scale eddies, while the dissipation is confined to the very small eddies (of size \(\eta\)). The question is how the energy transfers from the large scales to the small. Richardson suggested that this is a multistage process, involving a hierarchy of eddies from \(l\) down to \(\eta\). The idea is that large eddies break up into smaller ones, which in turn produce even smaller structures, and so on. (Figure 3.9(a).) This process is driven by inertia since the viscous forces are insignificant except at the smallest scales. So the rate \(\Pi\) at which energy is lost is

\[
\Pi \sim \frac{x^2}{l} \sim \frac{x^3}{l^2}
\]

Since the rate of destruction of energy at the small scales, \(\dot{e}\), must be equal to \(\Pi\) in statistically steady turbulence, and is still approximately equal to \(\Pi\) in decaying turbulence, we have

\[
\dot{e} = \Pi \sim \frac{\nu^2}{l}.
\]

In summary then, there are two things that we know about the small scales. First, they have a characteristic size, \(\eta\), and velocity, \(v\), such that \((\nu \eta / v) \sim 1\). (Smaller eddies would be rubbed out by viscosity while larger ones would not feel the viscous stresses.) Second, the dissipation at the small scales is

\[
\dot{e} = 2 \pi \rho \eta^2 \sim \nu^2 / l^2
\]

The term 'break-up' is being used rather loosely here. We describe the mechanism by which energy is passed to smaller scales in Chapter 1. For the present purposes we might think of an eddy as a blob or filament of vorticity, and imagine that the chaotic velocity...
If the turbulence is also isotropic, i.e., looks the same from all orientations, then the power spectrum depends only on the magnitude $k$ of the wavenumber, and we can partition the variance into different wavenumber bands:

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![Energy spectrum diagram](image-url)
by the average energy dissipation rate $\varepsilon$ per unit mass (units of energy per unit mass per unit time, or $m^2 s^{-3}$). If the largest scale eddies have characteristic eddy velocity $V$, dimensional analysis implies

$$\varepsilon \propto V^3/L$$

and the dissipation timescale is the eddy turnover timescale $L/V$ (which is typically $O(1000 \text{ m}/1 \text{ m s}^{-1}) = 1000 \text{ s}$ in the ABL). This means that if its large-scale energy source is cut off, turbulence decays within a few turnover times. The viscous dissipation lengthscale or Kolmogorov scale $\eta$ depends on $\varepsilon$ ($m^2 s^{-3}$) and $v$ ($m^2 s^{-1}$), so dimensionally

$$\eta = (\nu^3/\varepsilon)^{1/4} (\approx 1 \text{ mm for the ABL}) = \text{Re}^{-3/4} L$$

Kolmogorov argued that the energy spectrum $E(k)$ within the inertial subrange can depend only on the lengthscale, measured by wavenumber $k$, and $\varepsilon$. Noting that $E(k)$ has units of TKE/wavenumber $= m^2 s^{-2}/m^{-1} = m^3 s^{-2}$, dimensional analysis implies the famous $-5/3$ power law,

$$E(k) \propto \varepsilon^{-2/3} k^{-5/3}, \quad L^{-1} \ll k \ll \eta^{-1}$$

Similarly, the spatial power spectra of velocity components and scalars $a$ also follow $\hat{S}_a(k) \propto k^{-5/3}$ in the inertial range.

The spatial power spectrum can be measured in one direction by a sensor moving with respect to the boundary layer at a speed $U$ comparable to or larger than $V$, i.e. if the wind is blowing different turbulent eddies past a sensor on the ground, or if we take measurements from an aircraft. We must invoke Taylor’s (‘frozen turbulence’) hypothesis that the statistics of the turbulent field are similar to what we would measure if the turbulent field remained unchanged and just advected by with the mean speed $U$. In general, empirically this appears to be a good assumption. Temporal power spectra $\hat{S}_a(\omega)$ gathered in this way can be converted to spatial power spectral by substituting $\omega = Uk$,

$$\hat{S}_a(k) = U \hat{S}_a(Uk)$$

Thus, we expect an $\omega^{-5/3}$ temporal power spectrum for scalars and velocity components in the inertial subrange.

The figures below show measurements from a tethered balloon stationed in a convecting cloud-topped boundary layer at 85% of the inversion height. The mean wind of $U = 7 \text{ m s}^{-1}$ is considerable larger than the characteristic large-eddy velocity of $V = 1 \text{ m s}^{-1}$, so Taylor’s hypothesis is safe. The time series shows up and downdrafts associated with large eddies with width and height comparable to the BL depth of 1 km, with turbulent fluctuations associated with smaller eddies. The corresponding temporal power spectrum (triangles) is plotted as $\omega \hat{S}_a(\omega)$, as expected, this has a $\omega^{2/3}$ dependence in the inertial range, and decays at low frequencies that correspond to lengthscales larger than $L$.

The second spectrum (circles) is in the entrainment zone, which is in a very sharp and strong inversion (stable layer) at the BL top. Here, large scale, strong, vertical motions are suppressed, and the turbulence is highly anisotropic at these scales, but at small scales (a few meters or less) an inertial range is still observed.
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in the flow. Horizontal line and area averages are more meaningful, considering the inhomogeneity of the PBL in the vertical due to shear and buoyancy effects.

Finally, the type of averaging which is almost always used in theory, but rarely in practice, is the ensemble or probability mean. It is an arithmetical average of a very large (approaching infinity) number of realizations of a variable or a function of variables, which are obtained by repeating the experiment over and over again under the same general conditions. It is quite obvious that the ensemble averages would be nearly impossible to obtain under the varying weather conditions in the atmosphere, over which we have little or no control. Even in a controlled laboratory environment, it would be very time consuming to repeat an experiment many times to obtain ensemble averages.

The fact that different types of averages are used in theory and experiments requires that we should know about the conditions in which time or space averages might become equivalent to ensemble averages. It has been found that the necessary and sufficient conditions for the time and ensemble means to be equal are that the process (in our case, flow) be stationary (i.e., the averages be independent of time) and the averaging period very large ($T \to \infty$) for a more detailed discussion of these conditions, see Monin and Yaglom, 1971). It should suffice to say that these conditions cannot be strictly satisfied in the atmosphere so that the above-mentioned equivalence of averages can only be approximate. The corresponding conditions for the equivalence of spatial and ensemble averages are spatial homogeneity (independence of averages to spatial coordinates in one or more directions) and very large averaging paths or areas. These conditions are even more restrictive and difficult to satisfy in micrometeorological applications. Quasi-stationarity over a limited period of time, or quasi-homogeneity over a limited length or area in the horizontal plane, is the best one can hope for in an idealized PBL over a homogeneous surface under undisturbed weather conditions. Under such conditions, one might expect an apparent correspondence between time or space averages used in experiments and ensemble averages used in theories and mathematics.

The mean and fluctuating parts of variables in the atmospheric boundary layer depend on the type of variable, observation height relative to the PBL height, atmospheric stability, the type of surface, and other factors. In the surface layer, during undisturbed weather conditions, the magnitudes of vertical fluctuations, on the average, are much larger than the mean vertical velocity, the magnitudes of horizontal velocity fluctuations are of the same order or less than the mean horizontal velocity, and the magnitudes of the fluctuations in thermodynamic variables are at least two orders of magnitude smaller than their mean values. The relative magnitudes of turbulent fluctuations generally decrease with increasing stability and also with increasing height in the PBL. As an example, measured time series of velocity, temperature, and humidity fluctuations at a suburban site in Vancouver, Canada, during the daytime moderately unstable conditions are shown in Figure 8.1. These observations were taken at a height of 27.4 m above ground level, which falls within the unstable surface layer. The corresponding hourly averaged values of the variables were:

\[ U = 3.66 \text{ m s}^{-1}; \quad V = W = 0 \]
\[ T = 294.6 \text{ K}; \quad Q = 8.3 \text{ g m}^{-3} \]

Note the difference in the character of traces of horizontal and vertical velocity components, temperature, and absolute humidity fluctuations. Under unstable and convective conditions represented by these traces, buoyant updrafts and thermals cause strong asymmetry between positive and negative