Lecture 3. Turbulent fluxes and TKE budgets (Garratt, Ch 2)

The ABL, though turbulent, is not homogeneous, and a critical role of turbulence is transport and mixing of air properties, especially in the vertical. This process is quantified using ensemble averaging (often called Reynolds averaging) of the hydrodynamic equations.

Boussinesq Equations (G 2.2)

For simplicity, we will use the Boussinesq approximation to the Navier-Stokes equations to describe boundary-layer flows. This is quite accurate for the ABL (and ocean BLs as well), since:

1. The ABL depth O(1 km) is much less than the density scale height O(10 km).
2. Typical fluid velocities are O(1-10 m s^{-1}), much less than the sound speed.

The Boussinesq equations of motion are:

$$\frac{D\mathbf{u}}{Dt} + f\mathbf{k}\times\mathbf{u} = -\frac{\nabla p'}{\rho_0} + b\mathbf{k} + [\nu\nabla^2\mathbf{u}], \text{ where buoyancy } b = g\theta_v'/\theta_0$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\frac{D\theta}{Dt} = S_\theta + [\kappa\nabla^2\theta] , \qquad S_\theta \approx -\frac{1}{\rho_0 C_p}\frac{\partial R_N}{\partial z} \text{ in the absence of clouds}$$

$$\frac{Dq}{Dt} = S_q + [\kappa_q\nabla^2 q] \quad , S_q = 0 \text{ in the absence of precipitation}$$

Here p' is a pressure perturbation, θ is potential temperature, $q = q_v + q_l$ is mixing ratio (including water vapor q_v and liquid water q_l if present), and $\theta_v = \theta(1 + .608q_v - q_l)$ is virtual potential temperature including liquid water loading. S denotes source/sink terms, and ρ_0 and θ_0 are characteristic ABL density and potential temperature. κ and κ_q are the diffusivities of heat and water vapor. The most important source term for θ is divergence of the net radiative flux R_N (usually treated as horizontally uniform on the scale of the boundary layer, though this needn't be exactly true, especially when clouds are present). For noprecipitating BLs, $S_q = 0$.. For cloud-topped boundary layers, condensation, precipitation and evaporation can also be important.

Using mass continuity, the substantial derivative of any quantity a can be written in flux form

$$Da/Dt = \partial a/\partial t + \nabla\cdot(\mathbf{u}a).$$

Ensemble Averaging (G 2.3)

The ensemble average of Da/Dt is:

$$\overline{\frac{Da}{Dt}} = \frac{\overline{\partial a}}{\partial t} + \nabla\cdot\overline{\mathbf{u}a}$$

$$= \frac{\partial}{\partial t}\bar{a} + \frac{\partial}{\partial t}\overline{a'} + \frac{\partial}{\partial x}\overline{(\bar{u}+u')(\bar{a}+a')} + \frac{\partial}{\partial y}\overline{(\bar{v}+v')(\bar{a}+a')} + \frac{\partial}{\partial z}\overline{(\bar{w}+w')(\bar{a}+a')}$$

$$= \frac{\partial}{\partial t}\bar{a} + \nabla\cdot\bar{\mathbf{u}}\bar{a} + \frac{\partial}{\partial x}\overline{u'a'} + \frac{\partial}{\partial y}\overline{v'a'} + \frac{\partial}{\partial z}\overline{w'a'}$$

To summarize the rules of averaging:

$$
\begin{aligned}
\bar{c} &= c \\
\overline{(c\,A)} &= c\,\bar{A} \\
\overline{(\bar{A})} &= \bar{A} \\
\overline{(\bar{A}\,B)} &= \bar{A}\,\bar{B} \\
\overline{(A+B)} &= \bar{A} + \bar{B} \\
\overline{\left(\frac{dA}{dt}\right)} &= \frac{d\,\bar{A}}{dt}
\end{aligned}
\tag{2.4.2k}
$$

2.4.3 Reynolds Averaging

The averaging rules of the last section can now be applied to variables that are split into mean and turbulent parts. Let $A = \overline{A} + a'$ and $B = \overline{B} + b'$. Starting with the instantaneous value, A, for example, we can find its mean using the fifth and third rules of the previous section:

$$\overline{(A)} = \overline{(\overline{A} + a')} = \overline{(\overline{A})} + \overline{a'} = \overline{A} + \overline{a'}$$

The only way that the left and right sides can be equal is if

$$\overline{a'} = 0 \qquad (2.4.3a)$$

Another example: start with the product $\overline{B}\, a'$ and find its average. Employing the above result together with the fourth averaging rule, we find that

$$\overline{(\overline{B}\, a')} \;=\; \overline{B}\, \overline{a'} \;=\; \overline{B} \cdot 0 \;=\; 0 \qquad (2.4.3b)$$

Similarly, $\overline{\overline{A}\, b'} = 0$. One should not become too lax about the average of primed variables, as is demonstrated next.

The average of the product of A and B is

$$\begin{aligned}\overline{(A \cdot B)} &= \overline{(\overline{A} + a')\,(\overline{B} + b')} \\ &= \overline{(\overline{A}\,\overline{B} \;+\; a'\,\overline{B} \;+\; \overline{A}\,b' \;+\; a'\,b')} \\ &= \overline{(\overline{A}\,\overline{B})} + \overline{(a'\,\overline{B})} + \overline{(\overline{A}\,b')} + \overline{(a'\,b')} \\ &= \overline{A}\,\overline{B} \;+\; 0 \;+\; 0 \;+\; \overline{a'\,b'} \\ &= \overline{A}\,\overline{B} + \overline{a'\,b'} \end{aligned} \tag{2.4.3c}$$

The nonlinear product $\overline{a'\,b'}$ is NOT necessarily zero. The same conclusion holds for other nonlinear variables such as:

The three **eddy correlation** terms at the end of the equation express the net effect of the turbulence.

Consider a BL of characteristic depth H over a nearly horizontally homogeneous surface. The most energetic turbulent eddies in the boundary layer have horizontal and vertical lengthscale H and (by mass continuity) the same scale U for turbulent velocity perturbations in both the horizontal and vertical. The boundary layer structure, and hence the eddy correlations, will vary horizontally on characteristic scales $L_s >> H$ due to the impacf on the BL of mesoscale and synoptic-scale variability in the free troposphere. If we let {} denote 'the scale of', and assume $\{a'\} = A$, we see that **the vertical flux divergence is dominant**:

$$\left\{\frac{\partial}{\partial x}\overline{u'a'}\right\} = \frac{UA}{L_s} \ll \left\{\frac{\partial}{\partial z}\overline{w'a'}\right\} = \frac{UA}{H}$$

Thus (noting also that $\nabla \cdot \bar{\mathbf{u}} = 0$ to undo the flux form of the advection of the mean),

$$\boxed{\frac{\overline{Da}}{Dt} \approx \frac{\partial}{\partial t}\bar{a} + \bar{\mathbf{u}} \cdot \nabla\bar{a} + \frac{\partial}{\partial z}\overline{w'a'}}$$

If we apply this to the ensemble-averaged heat equation, and throw out horizontal derivatives of θ in the diffusion term using the same lengthscale argument $L_s >> H$ as above, we find

$$\boxed{\frac{\partial}{\partial t}\bar{\theta} + \bar{\mathbf{u}} \cdot \nabla\bar{\theta} = -\frac{\partial}{\partial z}(\overline{w'\theta'}) + \overline{S_\theta} + \left[\kappa\frac{\partial^2}{\partial z^2}\bar{\theta}\right]}$$

Thus, the effect of turbulence on $\bar{\theta}$ is felt through the convergence of the vertical eddy correlation, or **turbulent flux** of θ. The turbulent **sensible and latent heat fluxes** are the turbulent fluxes of θ and q in energy units of W m^{-2}:

$$\text{Turbulent sensible heat flux} = \rho_0 C_p \overline{w'\theta'}$$

$$\text{Turbulent latent heat flux} = \rho_0 L \overline{w'q'}$$

Except in the interfacial layer within mm of the surface, the diffusion term is negligible , so we've written it in square brackets.

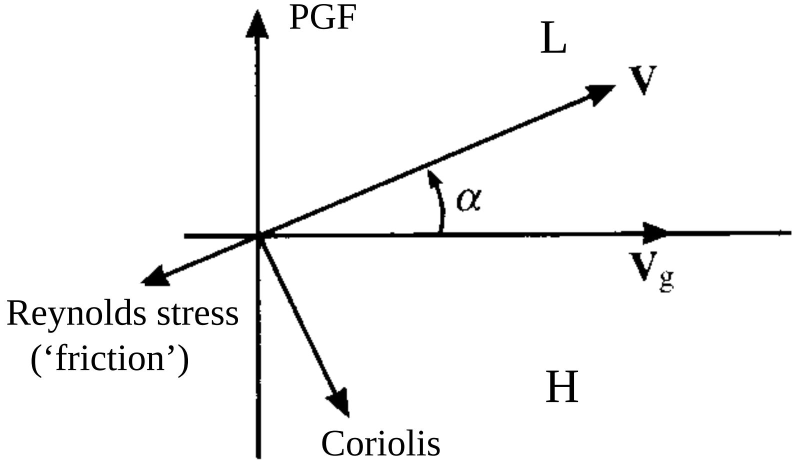
If geostrophic wind $\mathbf{u}_g$ is defined in the standard way, the ensemble-averaged momentum equations are

$$\frac{\partial \bar{u}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{u} = f(\bar{v} - v_g) - \frac{\partial}{\partial z}(\overline{u'w'})$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{v} = -f(\bar{u} - u_g) - \frac{\partial}{\partial z}(\overline{v'w'})$$

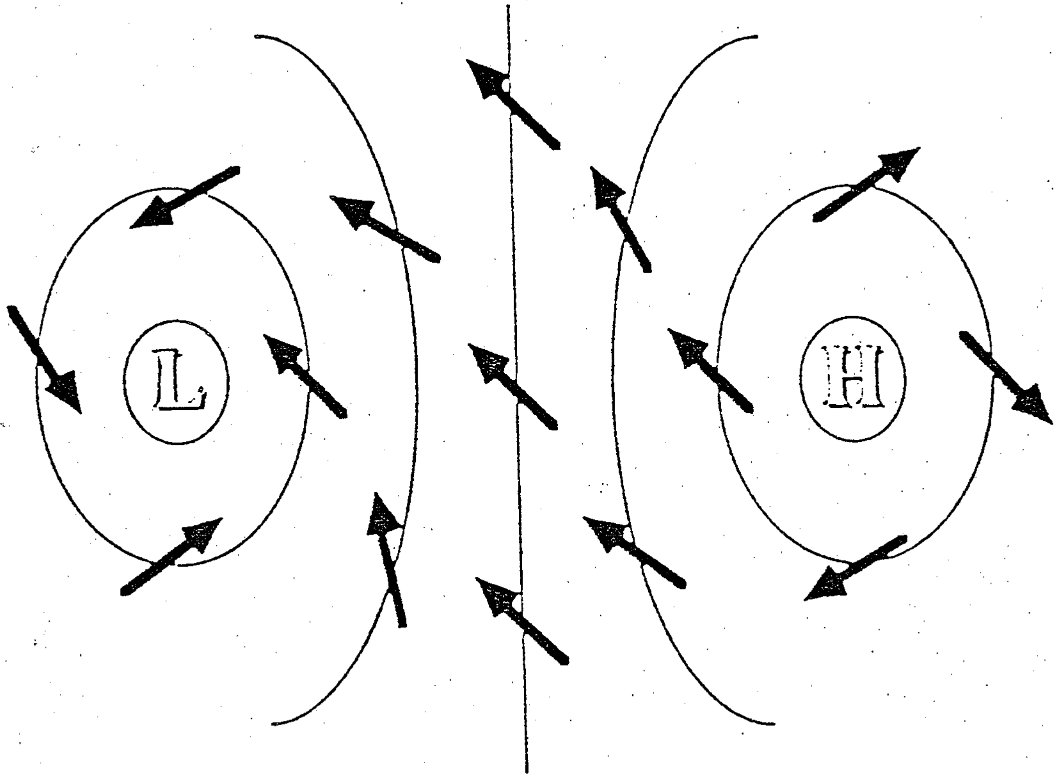
Often, but not always, the tendency and advection terms are much smaller than the two terms on the right hand side, and there is an approximate three-way force balance (see figure below) between momentum flux convergence, Coriolis force and pressure gradient force in the ABL such that the mean wind has a component down the pressure gradient. The **cross-isobar flow angle** α is the angle between the actual surface wind and the geostrophic wind.

If the mean profiles of actual and geostrophic velocity can be accurately measured, the momen-



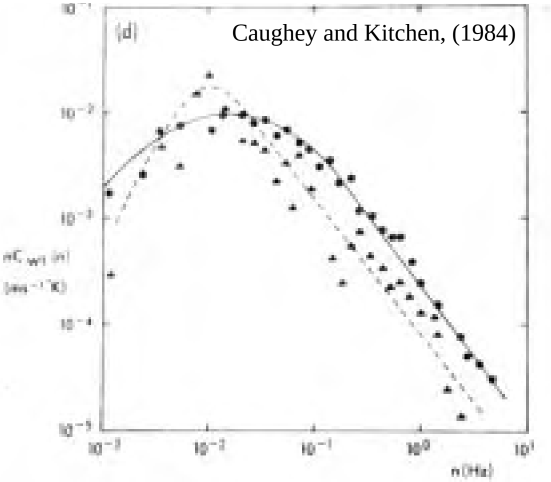
Surface layer force balance in a steady state BL ($f > 0$). Above the surface layer, the force balance is similar but the Reynolds stress need not be along -**V**.

tum flux convergence can be calculated as a residual in the above equations, and vertically integrated to deduce momentum flux. This technique was commonly applied early in this century, before fast-response, high data rate measurements of turbulent velocity components were perfected. It was not very accurate, because small measurement errors in either $\mathbf{u}$ or $\mathbf{u}_g$ can lead to large relative errors in momentum flux.



L
H

In most BLs, the vertical fluxes of heat, moisture and momentum are primarily carried by large eddies with lengthscale comparable to the boundary layer depth, except near the surface where smaller eddies become important. The figure below shows the **cospectrum** of w' and T', which is the Fourier transform of $w'T'$, from tethered balloon measurements at two heights in the cloud-topped boundary layer we plotted in the previous lecture. The cospectrum is positive, i. e. positive correlation between w' and T', at all frequencies, typical of a convective boundary layer. Most of the covariance between w' and T' is at the same low frequencies $n = \omega/2\pi \sim 10^{-2}$ Hz that had the maximum energy. Since the BL is blowing by the tethered balloon at the mean wind speed $\overline{U} = 7$ m s^{-1}, this frequency corresponds to large eddies of wavelength $\lambda = \overline{U}/n = 700$ m, which is comparable to the BL depth of 1 km.



Cospectrum of w' and T' at cloud base (triangles), top (circles) in convective BL.

Turbulent Energy Equation (G 2.5,6)

To form an equation for TKE $\bar{e} = \overline{\mathbf{u}' \cdot \mathbf{u}'}/2$, we dot $\mathbf{u}$ into the momentum equation, and take the ensemble average. After considerable manipulation, we find that for the nearly horizontally homogeneous BL ($H << L_s$),

$$\boxed{\frac{\partial}{\partial t}\bar{e} + \bar{\mathbf{u}} \cdot \nabla \bar{e} = S + B + T + D}$$

where

$$S = -\overline{u'w'}\frac{\partial \bar{u}}{\partial z} - \overline{v'w'}\frac{\partial \bar{v}}{\partial z} \quad \text{(shear production)}$$

$$B = \overline{w'b'} \quad \text{(buoyancy flux)}$$

$$T = -\frac{\partial}{\partial z}\left(\overline{w'e'} + \frac{1}{\rho_0}\overline{w'p'}\right) \quad \text{(transport and pressure work)}$$

$$D = -\nu\overline{|\nabla \times \mathbf{u}|^2} \quad \text{(dissipation, always negative, } -\varepsilon \text{ in Garratt)}$$

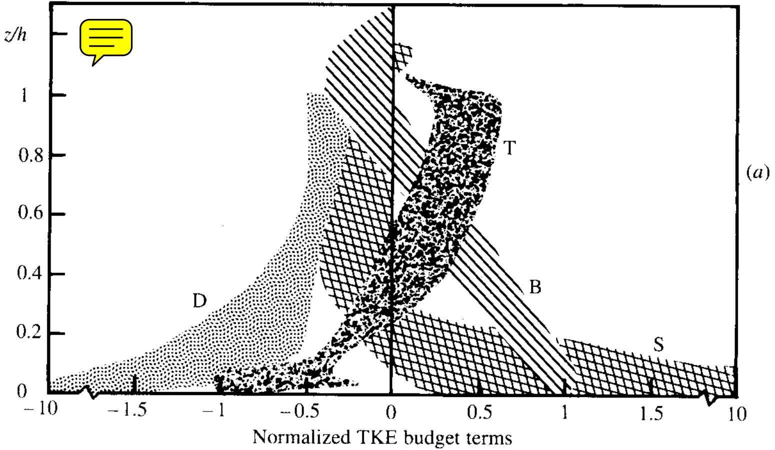
Shear production of TKE occurs when the momentum flux is downgradient, i. e. has a component opposite (or 'down') the mean vertical shear. To do this, the eddies must tilt into the shear. Kinetic energy of the mean flow is transferred into TKE. Buoyancy production of TKE occurs where relatively buoyant air is moving upward and less buoyant air is moving downward. Gravitational potential energy of the mean state is converted to TKE. Both S and B can be negative at some or all levels in the BL, but together they are the main source of TKE, so the vertical integral of $S + B$ over the BL is always positiveThe transport term mainly fluxes TKE between different levels, but a small fraction of TKE can be lost to upward-propagating internal gravity waves excited by turbulence perturbing the BL top. The dissipation term is the primary sink of TKE, and formally is related to enstrophy. In turbulent flows, the enstrophy is dominated by the *smallest* (dissipation) scales, so D can be considerable despite the smallness of ν.
Usually, the left hand side (the 'storage' term) is smaller than the dominant terms on the right hand side. The figure on the next page shows typical profiles of these terms for a daytime convectively driven boundary layer and a nighttime shear-driven boundary layer. In the convective boundary layer, transport is considerable. Its main effect is to homogenizing TKE in the vertical . With vertically fairly uniform TKE, dissipation is also uniform, except near the ground where it is enhanced by the surface drag . Shear production is important only near the ground (and sometimes at the boundary layer top). In the shear-driven boundary layer, transport and buoyancy fluxes are small everywhere, and there is an approximate balance between shear production and dissipation.

The **flux Richardson number**

$$\boxed{\mathrm{Ri}_f = - B/S}$$

characterizes whether the flow is stable ($\mathrm{Ri}_f > 0$), neutral ($\mathrm{Ri}_f \approx 0$), or unstable ($\mathrm{Ri}_f < 0$).

Garratt



z/h
1
0.8
0.6
0.4
0.2
0
D
T
B
S
(a)
−10
−1.5
−1
−0.5
0
0.5
1
1.5
10
Normalized TKE budget terms

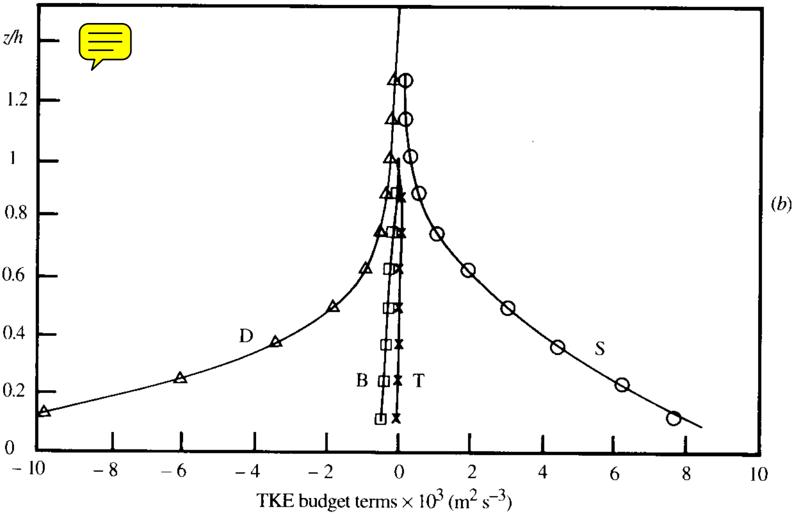


Fig. 2.4 Terms in the TKE equation (2.74b) as a function of height, normalized in the case of the clear daytime ABL (*a*) through division by $w_*{}^3/h$; actual terms are shown in (*b*) for the clear night-time ABL. Profiles in (*a*) are based on observations and model simulations as described in Stull (1988; Figure 5.4), and in (*b*) are from Lenschow *et al.* (1988) based on one aircraft flight. In both, B is the buoyancy term, D is dissipation, S is shear generation and T is the transport term. Reprinted by permission of Kluwer Academic Publishers.