Entraining BL (Bretherton Lect 12)

$$\frac{\Delta b(t)}{b_m} = \text{constant} \qquad (\text{similarity})$$

$\Delta b + b_m = N^2 h$, so

$$\boxed{\Delta b(t) = c\, N^2 h(t)} \qquad (c:\ \text{unknown})$$

What is c?

From (1): $+w_e \Delta b = +\beta B_0 \quad (\beta = 0.2)$

$w_e = \frac{dh}{dt}$ (if $\bar{w} = 0$) so combine

$$\boxed{\frac{dh}{dt} c\, N^2 h(t) = \beta B_0.}$$

Integrate from time 0 to t, with $h(0) = 0$:

$$\boxed{\frac{h^2}{2} c\, N^2 = \beta B_0 t.} \qquad (3)$$

From (2), we have $A = B_0 t$.
In this case, mechanical entrainment contributes to 'area' under initial profile of b, so $A = A_P - A_N$ (see diagram),

$b_m = (1-c) N^2 h$.

$A_P = b_m \left(\frac{b_m}{N^2}\right) \frac{1}{2}$ $\qquad$ use $z = \frac{b_m}{N^2}$

$A_N = \Delta b \left(\frac{\Delta b}{N^2}\right) \frac{1}{2}$

So (2) can be written

$$B_0 t = A = A_P - A_N$$

$$= \frac{b_m^2}{2N^2} - \frac{\Delta b^2}{2N^2}$$

$$= \frac{(1-c)^2 - c^2}{2} N^2 h^2 \qquad \begin{cases} \Delta b = c N^2 h \\ b_m = (1-c) N^2 h \end{cases}$$

$$= \frac{(1-2c)}{2} N^2 h^2 \quad (4) \qquad (1-c)^2 = 1 - 2c + c^2$$

Divide (3) by (4):

$$\frac{\beta B_0 t}{B_0 t} = \frac{h^2 c N^2 / 2}{(1-2c) N^2 h^2 / 2}$$

$$\beta = \frac{c}{1-2c}, \text{ or } c = \frac{\beta}{1+2\beta}.$$

Solve (4) for h:

$$B_0 t = (1-2c) N^2 h^2 / 2 \quad \text{so}$$

$$h_{entr} = \left[\frac{2 B_0 t}{N^2 (1-2c)}\right]^{1/2} = \left[\frac{2 B_0 t (1+2\beta)}{N^2}\right]^{1/2}$$

$$\approx h_{encr} (1+2\beta)^{1/2} \qquad \frac{1}{1-2c} = 1+2\beta$$

$$\approx h_{encr} (1+\beta) \quad \text{for } \beta \ll 1.$$

Use $h_{encr} = \left(\frac{2 B_0 t}{N^2}\right)^{1/2}$.

$$(1+\beta)^2 = 1 + 2\beta + \beta^2 \approx 1 + 2\beta \quad \text{for } \beta \ll 1.$$

square roots:

$$(1+\beta) \approx (1+2\beta)^{1/2}$$