Length scales in turbulent flows

- Large eddies do most of the transport of momentum and scalars.
- At very small scales, viscosity can smooth out velocity fluctuations, and dissipate small-scale kinetic energy into heat.
- Small-scale motion depends only on rate of kinetic energy supply by large-scale motion and on $\nu$.
- Assume that rate of energy supply = rate of dissipation: Kolmogorov's "equilibrium theory" of small-scale structure.

- Governing parameters:
  \[ \varepsilon \text{ (m}^2 \text{s}^{-3}) \]
  \[ \eta \text{ (m}^2 \text{s}^{-1}) \]

  **Kolmogorov microscales:**
  \[ n \equiv (\nu^3/\varepsilon)^{1/4} \text{ length} \]
  \[ \tau \equiv (\nu/\varepsilon)^{1/2} \text{ time} \]
  \[ v \equiv (\nu\varepsilon)^{1/4} \text{ velocity} \]

  **Reynolds number:**
  \[ \frac{UL}{V} = \frac{\eta V}{V} = 1 \]

  **Dissipation rate:**
  \[ \varepsilon = \nu \left( \frac{V}{n} \right)^2 = \nu \frac{1}{\tau^2} \]
Viscous Dissipation Rate

Stull (section 4.3.1) derived the prognostic equation for velocity variance (see Eq. 4.3.1g):

\[
\frac{\partial u'_i^2}{\partial t} = \ldots - 2\varepsilon
\]

where

\[
\varepsilon = \nu \left( \frac{\partial u'_i}{\partial x_j} \right)^2 = \nu \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j}.
\]
Dissipation Rate Estimate

\[ \frac{dk}{dt} \sim E \sim k \frac{K}{T_0} \]

K: K.E. per unit mass
\( T_0 \): large-scale turbulence

\[ K \sim U_0^3 \]

\[ T_0 \sim \frac{l_0}{U_0} \]

\( l_0 \): size of largest eddies

Rate at which large eddies supply k.e. to small eddies \( \sim \frac{1}{T_0} \). Then \( E \sim U_0^2 \cdot \frac{U_0}{l_0} = \frac{U_0^3}{l_0} \).

The result is an energy cascade from the largest eddies to smaller and smaller eddies until the eddy sizes are so small that viscous dissipation is almost immediate.

This is captured by Lewis Richardson's poem (1922):

Big whirls have little whirls,
Which feed on their velocity;
And little whirls have lesser whirls,
And so on to viscosity.

[show diagram of energy spectrum]
Figure 3.9  (a) A schematic representation of the energy cascade (after Frisch). (b) The energy cascade in terms of energy versus wave number.
Ratios of smallest to largest scales
one obtained from Kolmogorov scales and
$\varepsilon \sim \frac{U_0^3}{l_0}$:

$\frac{\eta}{l_0} \sim Re^{-3/4}$

$\frac{\nu}{U_0} \sim Re^{-1/4}$

$\frac{\tau}{T_0} \sim Re^{-1/2}$

At high $Re$, velocity and timescales of smallest
eddies ($\eta, \tau$) are small compared to those of the
largest eddies ($U_0, T_0$).

The ratio $\frac{\eta}{l_0}$ decreases with increasing $Re$,
(see Fig. 1.15, [Davidson]). As a consequence,
at sufficiently high $Re$, there is a range of
scales $l$ such that $l_0 \gg l \gg \eta$. Because
$Re = \frac{\nu u(l)}{\nu}$ is large, viscosity is not important,
and the statistics of motions of scale $l$ have
a universal form uniquely determined by $\varepsilon$
indeedent of $l$. This range of scales is
the inertial subrange. The remainder of
the universal equilibrium range is the
dissipation range. See Fig. 6.1 (Pope).
Figure 1.15 The influence of Re on the smallest scales in a turbulent wake. Note that the smallest eddies are much smaller in the high-Re flow. (After Tennekes and Lumley 1972.)
Fig. 6.1. Eddy sizes $\ell$ (on a logarithmic scale) at very high Reynolds number, showing the various length scales and ranges.
Given an eddy size \( l \) (in the inertial subrange), characteristic scales \( u(l) \), \( \tau(l) \) can be formed from \( \varepsilon \) and \( l \):

\[
\begin{align*}
u(l) &= (\varepsilon l)^{1/3} = \sqrt{\frac{\varepsilon}{n}} \approx u_0 \left( \frac{l}{l_0} \right)^{1/3} \\
\tau(l) &= (l^2/\varepsilon)^{1/3} = \tau \left( \frac{l}{l_0} \right)^{2/3} \approx T_0 \left( \frac{l}{l_0} \right)^{2/3}
\end{align*}
\]

\( u(l) \) and \( \tau(l) \) decrease as \( l \) decreases. Note that

\[
\frac{u(l)^2}{\tau(l)} = \varepsilon.
\]

This suggests that the rate at which energy is transferred from eddies larger than \( l \) to those smaller than \( l \) is independent of \( l \) (for \( l \) in the inertial subrange) and equal to \( \varepsilon \). See Fig. 6.2 (Pope).

The rate of energy transfer from large scales determines the rate in the inertial subrange, and hence the rate that enters the dissipation range, and hence the dissipation rate \( \varepsilon \). See Fig. 6.2 (Pope).
Fig. 6.2. A schematic diagram of the energy cascade at very high Reynolds number.
Estimate $e, n, u$, and $cu$ cloud

$L = 500 \text{ m}$
$u = 1 \text{ m/s}$
$e = u^3/l = 1/500 \text{ m}^2 \text{s}^{-3} = 0.002 \text{ m}^2 \text{s}^{-3}$
$n = (u^3/e)^{1/4} = 1 \text{ mm}$
$v = 15 \times 10^{-6} \text{ m}^2 \text{s}^{-1}$

Total dissipation rate:

$\text{cloud volume} = (500 \text{ m})^3$
$\text{cloud mass} = (500 \text{ m})^3 \times 1 \text{ kg/m}^3$
$= 1.25 \times 10^8 \text{ kg}$

Total dissipation:
$0.002 \text{ m}^2 \text{s}^{-3} \times 1.25 \times 10^8 \text{ kg}$
$= 2.50 \times 10^2 \text{ kW}$