Length scales in turbulent flows

- Large eddies do most of the transport of momentum and scalars.
- At very small scales, viscosity can smooth out velocity fluctuations, and dissipate small-scale kinetic energy into heat.
- Small-scale motion depends only on rate of kinetic energy supply by large-scale motion and on ν.
- Assume that rate of energy supply = rate of dissipation: Kolmogorov's "equilibrium theory" of small-scale structure.

- Governing parameters:

$$\varepsilon \ (m^2 s^{-3})$$
$$\nu \ (m^2 s^{-1})$$

Kolmogorov microscales:

$\eta \equiv (\nu^3/\varepsilon)^{1/4}$ length

$\tau \equiv (\nu/\varepsilon)^{1/2}$ time

$v \equiv (\nu\varepsilon)^{1/4}$ velocity

Reynolds number: $\dfrac{UL}{\nu} \sim \dfrac{\eta v}{\nu} = 1$

Dissipation rate: $\varepsilon = \nu\left(\dfrac{v}{\eta}\right)^2 = \nu \dfrac{1}{\tau^2}$

Viscous Dissipation Rate

Stull (section 4.3.1) derived the prognostic equation for velocity variance (see Eq. 4.3.1g):

$$\frac{\partial \overline{u_i'^2}}{\partial t} = \ldots - 2\varepsilon$$

where

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} = \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}.$$

Dissipation Rate Estimate

$\frac{dK}{dt} \sim \varepsilon \sim \frac{K}{T_0}$ K: K.E. per unit mass in large-scale turbulence

$K \sim u_0^2$

$T_0 \sim l_0/u_0$

l_0: size of largest eddies

Rate at which large eddies supply K.E. to small eddies $\sim 1/T_0$. Then $\varepsilon \sim u_0^2 \cdot u_0/l_0 = u_0^3/l_0$

This est. claims that large eddies lose a significant fraction of their K.E. in one "turnover" time, l_0/u_0.

The result is an energy cascade from the largest eddies to smaller and smaller eddies until the eddy sizes are so small that viscous dissipation is almost immediate.

This is captured by Lewis Richardson's poem (1922):

Big whorls have little whorls,
which feed on their velocity;
And little whorls have lesser whorls,
and so on to viscosity.

[show diagram of energy spectrum]

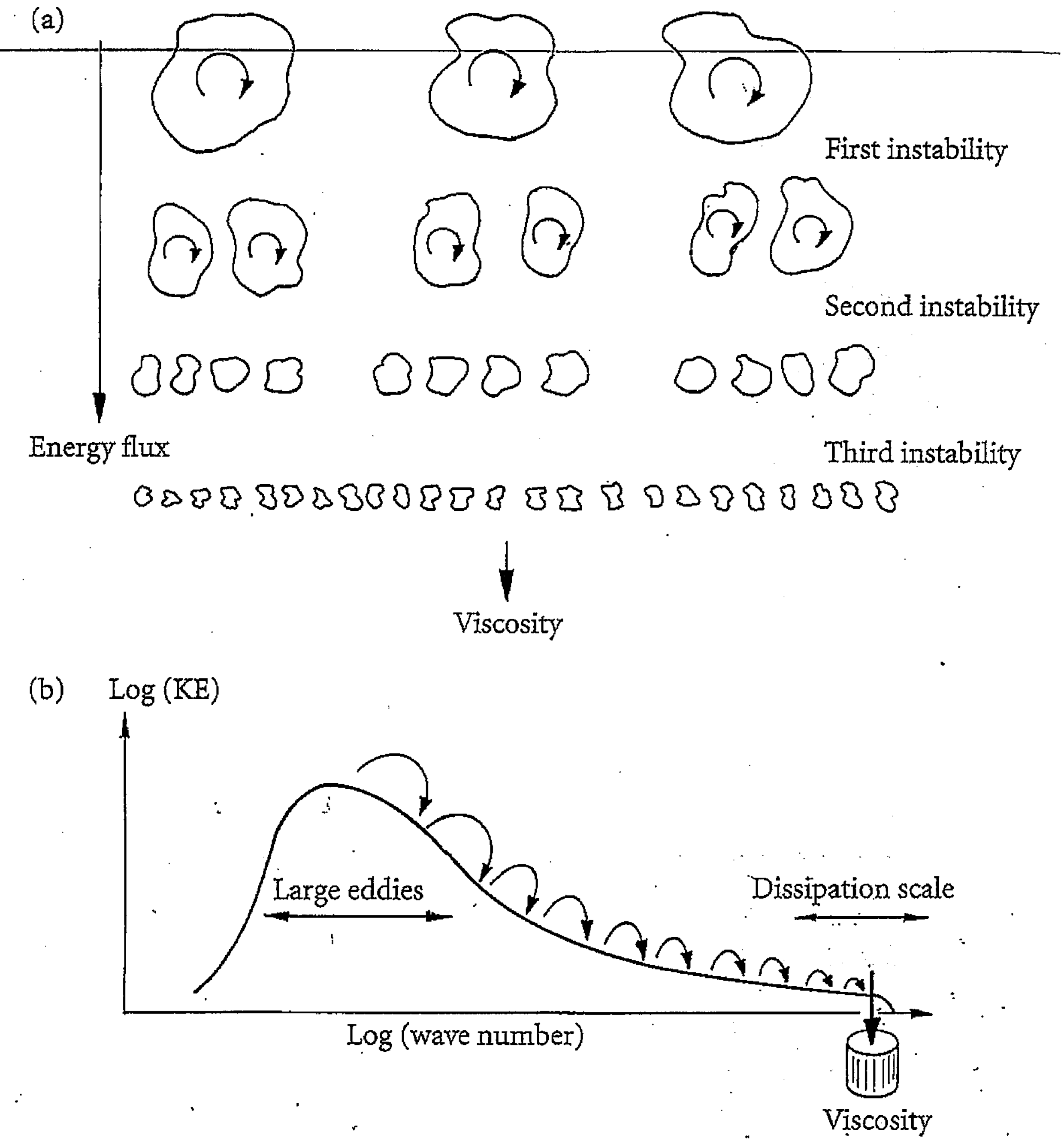

Figure 3.9 (a) A schematic representation of the energy cascade (after Frisch). (b) The energy cascade in terms of energy versus wave number.

Ratios of smallest to largest scales are obtained from Kolmogorov scales and $\varepsilon \sim \frac{u_0^3}{l_0}$:

$$\frac{\eta}{l_0} \sim Re^{-3/4} \qquad Re = \frac{u_0 l_0}{\nu}$$

$$\frac{v}{u_0} \sim Re^{-1/4}$$

$$\frac{\tau}{T_0} \sim Re^{-1/2}$$

At high Re, velocity and timescales of smallest eddies (v, τ) are small compared to those of the largest eddies (u_0, T_0).

The ratio η / l_0 decreases with increasing Re. (See Fig. 1.15, [Davidson]). As a consequence, at sufficiently high Re, there is a range of scales l such that $l_0 \gg l \gg \eta$. Because $Re = \frac{l\, u(l)}{\nu}$ is large, viscosity is not important, and the statistics of motions of scale l have a universal form uniquely determined by ε, independent of ν. This range of scales is the <u>inertial</u> <u>subrange</u>. The remainder of the <u>universal</u> <u>equilibrium</u> <u>range</u> is the <u>dissipation</u> <u>range</u>. See <u>Fig. 6.1</u> (Pope).

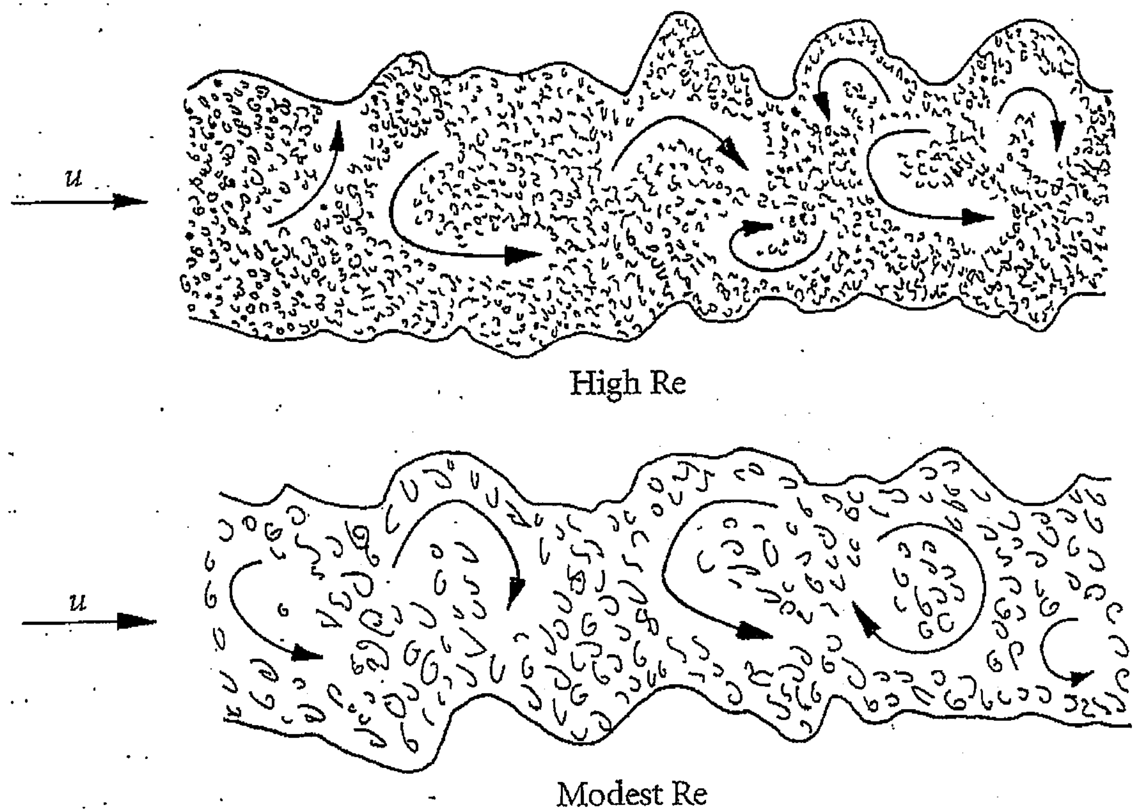

Figure 1.15 The influence of Re on the smallest scales in a turbulent wake. Note that the smallest eddies are much smaller in the high-Re flow. (After Tennekes and Lumley 1972.)

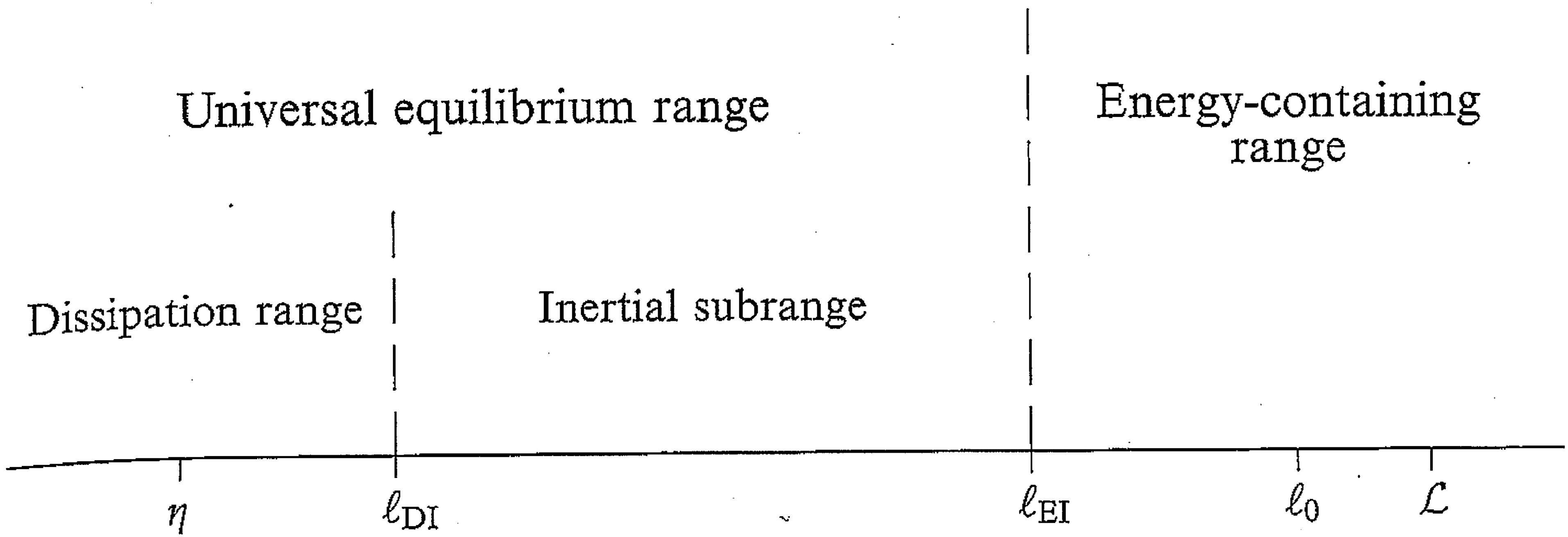

Fig. 6.1. Eddy sizes ℓ (on a logarithmic scale) at very high Reynolds number, showing the various lengthscales and ranges.

Given an eddy size l (in the inertial subrange), characteristic scales $u(l)$, $\tau(l)$ can be formed from ε and l:

$$u(l) = (\varepsilon l)^{1/3} = v\left(\frac{l}{\eta}\right)^{1/3} \approx u_0\left(\frac{l}{l_0}\right)^{1/3}$$
$$\tau(l) = (l^2/\varepsilon)^{1/3} = \tau\left(\frac{l}{\eta}\right)^{2/3} \approx T_0\left(\frac{l}{l_0}\right)^{2/3}$$

$u(l)$ and $\tau(l)$ decrease as l decreases.
Note that

$$\frac{u(l)^2}{\tau(l)} = \varepsilon .$$

This suggests that the rate at which energy is transferred from eddies larger than l to those smaller than l is independent of l (for l in the inertial subrange) and equal to ε. See Fig. 6.2 (Pope).

The rate of energy transfer from large scales determines the rate in the inertial subrange, and hence the rate that enters the dissipation range, and hence the dissipation rate ε. See Fig 6.2 (Pope).

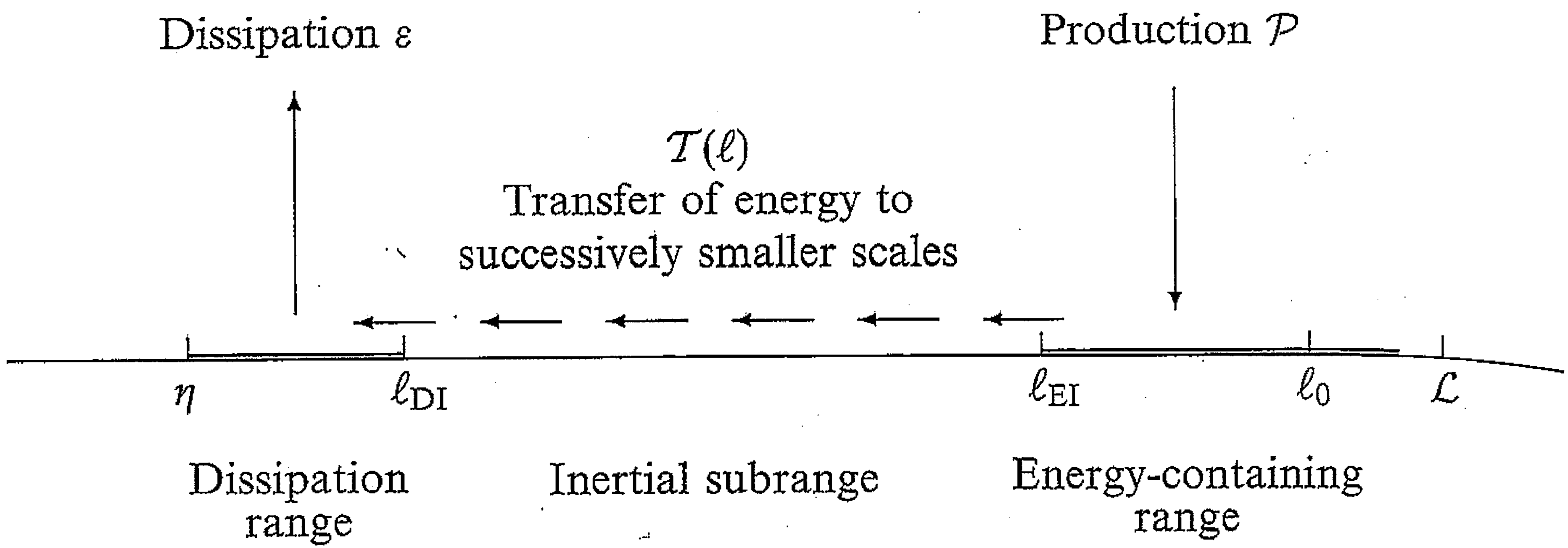

Fig. 6.2. A schematic diagram of the energy cascade at very high Reynolds number.

Estimate ε, η in a cu cloud

$L \sim 500$ m

$u \sim 1$ m/s

$\varepsilon \sim u^3/l = 1/500 \; m^2 s^{-3} \sim 0.002 \; m^2 s^{-3}$

$\eta \sim (\nu^3/\varepsilon)^{1/4} = 1$ mm $\qquad \nu = 15 \times 10^{-6} \; m^2 s^{-1}$

Total dissipation rate:

cloud volume $= (500 \; m)^3$

cloud mass $= (500 \; m)^3 \times 1$ kg/m3

$= 1.25 \times 10^8$ kg

Total dissipation $= .002 \; m^2 s^{-3} \times 1.25 \times 10^8$ kg

$= 2.50 \times 10^2$ kW