PBL Momentum Eqs. (Holton 5.3)

B.L. approx. + neglect of viscosity:

\[ \frac{\overline{D\bar{u}}}{Dz} = -\frac{1}{\rho_0} \frac{d\bar{p}}{dx} + f\bar{v} - \frac{\partial \bar{u}'}{\partial z} \]  

(1)

\[ \frac{\overline{D\bar{v}}}{Dz} = -\frac{1}{\rho_0} \frac{d\bar{p}}{dy} - f\bar{u} - \frac{\partial \bar{v}'}{\partial z} \]  

(2)

Unknowns: \( \bar{u}'w' \), \( \bar{v}'w' \)

(Assume \( \delta\bar{p}/\delta x, \delta\bar{p}/\delta y \) known.)

Approx. mid-lat. balance:

\[-f(\bar{v}' - \bar{v}_g) = -\frac{\partial \bar{u}'w'}{\partial z} = 0 \]  

(3)

\[-f(\bar{u}' - \bar{u}_g) - \frac{\partial \bar{v}'w'}{\partial z} = 0 \]  

(4)

where  \( -f\bar{v}_g = -\frac{1}{\rho_0} \frac{\delta\bar{p}}{\delta x} \),  \( f\bar{u}_g = -\frac{1}{\rho_0} \frac{\delta\bar{p}}{\delta y} \).

Mixed Layer Model

CBL is well-mixed:

\[ \begin{align*}
\text{FA} & \quad \text{EZ} \\
\text{ML} & \quad \text{SL}
\end{align*} \]
PBL Momentum Eqs. (Holton 5.3)

B.I.L. approx. + neglect of viscosity :

\[
\frac{D \overline{u}}{Dt} = -\frac{1}{\rho_0} \frac{d \overline{p}}{dx} + \overline{f u} - \frac{d \overline{u} u'}{dz} \tag{1}
\]

\[
\frac{D \overline{v}}{Dt} = -\frac{1}{\rho_0} \frac{d \overline{p}}{dy} - \overline{f u} - \frac{d \overline{v} u'}{dz} \tag{2}
\]

Unknowns: \( \overline{u'} \), \( \overline{v'} \)

(Assume \( \overline{p}/\partial x, \overline{p}/\partial y \) known.)

Approx. mid-lat. balance :

\[
f (\overline{v} - \overline{v_g}) = -\frac{d \overline{u' v'}}{dz} = 0 \tag{3}
\]

\[-f (\overline{u} - \overline{u_g}) - \frac{d \overline{v' u'}}{dz} = 0 \tag{4}
\]

where \(-f \overline{v_g} = -\frac{1}{\rho_0} \frac{d \overline{p}}{\partial x}, \quad f \overline{u_g} = -\frac{1}{\rho_0} \frac{d \overline{p}}{\partial y}\)

Mixed Layer Model

CBL is well-mixed:

\[\theta\]

\[\overline{u}\]
occurrence: over land during sunny days, over oceans when $ST > w_T$.

ML model:
- $\bar{\Theta}(z), \bar{u}(z), \bar{v}(z)$ const.
- Fluxes must be linear in $z$.
- Assume turbulence ($\Rightarrow$ fluxes) $\rightarrow 0$ at BL top.

Linear flux profile:
$$F(z) = F_s (1 - z/h).$$

Need $F_s$.

Use bulk aerodynamic formula (observationally based):
$$\frac{(\bar{u}'\bar{w}')_s}{h} = -Cd_{\text{drag}} \frac{\bar{V}}{u} \bar{u} \quad (5)$$
$$\frac{(\bar{v}'\bar{w}')_s}{h} = -Cd_{\text{drag}} \frac{\bar{V}}{u} \bar{v} \quad (6)$$

$Cd$: drag coeff. (non-dim.) $\approx 1.5 \times 10^{-3}$ over oceans; much larger over rough ground.

Can now integrate (3), (4) using (5), (6):
From surface to $z=h$:
$$f(\bar{v} - \bar{v}_g) = \frac{(\bar{u}'\bar{w}')_s}{h} = Cd_{\text{drag}} \frac{\bar{V}}{u} \bar{u}/h \quad (7)$$
$$f(\bar{u} - \bar{u}_g) = \frac{(\bar{v}'\bar{w}')_s}{h} = Cd_{\text{drag}} \frac{\bar{V}}{u} \bar{v}/h \quad (8)$$
occurrence: over land during sunny days, over oceans when $ST > m \bar{T}_s$.

M.L. model:
- $\bar{\theta}(z), \bar{u}(z), \bar{v}(z)$ const.
- Fluxes must be linear in $z$.
- Assume turbulence ($\Rightarrow$ fluxes $\to 0$ at BL top).

Linear flux profile:
$F(z) = F_s \left(1 - \frac{z}{h}\right)$.

Need $F_s$.
Use bulk aerodynamic formula (observationally based):

$$\langle u'w' \rangle_s = -C_d \frac{\bar{v}}{\bar{u}} |\bar{u}|$$  \hspace{1cm} (5)

$$\langle v'w' \rangle_s = -C_d \frac{\bar{v}}{\bar{u}} |\bar{v}|$$  \hspace{1cm} (6)

$C_d$: drag coeff, (non-dim.) $\approx 1.5 \times 10^{-3}$
over oceans: much larger over rough ground.

Can now integrate (3), (4) using (5), (6):
From surface to $z=h$:

$$f(\bar{v} - \bar{v}_g) = -\frac{\langle u'w' \rangle_s}{\bar{u}} = C_d \frac{\bar{v}}{\bar{u}} |\bar{u}| h$$  \hspace{1cm} (7)

$$f(\bar{u} - \bar{u}_g) = -\frac{\langle v'w' \rangle_s}{\bar{v}} = C_d \frac{\bar{v}}{\bar{v}} |\bar{v}| h$$  \hspace{1cm} (8)
Choose axes so $\vec{g} = 0 \Rightarrow$

$$\vec{v} = x_s \sqrt{\frac{\vec{v}}{\vec{u}}}$$
$$\vec{u} = \vec{u}_g - x_s \sqrt{\frac{\vec{v}}{\vec{u}}}$$

where

$$x_s \equiv \frac{cd}{c fh}.$$ 

Wind speed $|\vec{v}| = (\vec{u}^2 + \vec{v}^2)^{1/2}$ is less than $|\vec{v}_g| = \vec{u}_g$, and vector balance gives cross-isobar flow toward lower $p$.

Ex.

$$\vec{u}_g = 10 \text{ m/s},$$

$$x_s = 0.05 \text{ m}^{-1} \text{ s},$$

$$\vec{u} = 8.28 \text{ m/s},$$

$$\sqrt{\frac{\vec{v}}{\vec{u}}} = 3.77 \text{ m/s},$$

$$|\vec{v}| = 9.10 \text{ m/s}.$$ 

Work is done by $\vec{F}_g$ to balance KE loss due to friction.

Balance

$$\vec{F}_T$$

$$}\vec{c}_0$$

$$\vec{v}$$

$$\vec{u}$$

$$\rho$$

$$\rho - 2 \delta \rho$$

$$\rho - \delta \rho$$

$$\rho$$
The Atmospheric Boundary Layer

unstably stratified boundary layer is the *Deardorff* velocity scale

\[ w_s = \left[ \frac{g \cdot z_i \cdot \bar{w} \cdot \bar{\theta}_s}{T_v} \right]^{1/3} \]  

(9.13)

where \( z_i \) is the depth of the boundary layer and the subscript \( s \) denotes at the surface. Values of \( w_s \) have been determined from field measurements and numerical simulations under a wide range of conditions. Typical magnitudes of \( w_s \) are \( \sim 1 \text{ m s}^{-1} \), which corresponds to the average updraft velocities of thermals.

Another scale \( u_s \), the *friction velocity*, is most applicable to statically neutral conditions in the surface layer, within which the turbulence is mostly mechanically generated. It is given by

\[ u_s = \left[ \frac{u'w'^2 + v'w'^2}{4} \right]^{1/4} = \left( \frac{\tau_i}{\rho} \right)^{1/2} \]  

(9.14)

where \( \rho \) is air density, \( \tau_i \) is *stress* at the surface (i.e., drag force per unit surface area), and covariances \( u'w' \) and \( v'w' \) are the *kinematic momentum fluxes* (vertical fluxes of \( u \) and \( v \) horizontal momentum, respectively).

The altitude of the capping inversion, \( z_i \), is the relevant length scale for the whole boundary layer for statically unstable and neutral conditions. Within the bottom 5% of the boundary layer (referred to as the *surface layer*), an important length scale is the *aerodynamic roughness length*, \( z_0 \), which indicates the roughness of the surface (see Table 9.2). For statically nonneutral conditions in the surface layer, there is an additional length scale, called the *Obukhov length*

\[ L = \frac{-u_s^3}{k \cdot (g/T_v) \cdot \left( \bar{w} \cdot \bar{\theta}_s \right)_s} \]  

(9.15)

where \( k = 0.4 \) is the von Karman constant. The absolute value of \( L \) is the height below which mechanically generated turbulence dominates.

Typical timescales for the convective boundary layer and the neutral surface layer are

\[ t_s = \frac{z_i}{w_s} \quad t_{sSL} = \frac{z}{u_s} \]  

(9.16)

where \( z \) is height above the surface.

For the convective boundary layer, \( t_s \) is of order 15 min, which corresponds to the turnover time for the largest convective eddy circulations, which extend from the Earth’s surface all the way up to the capping inversion.

In summary, for convective boundary layers (i.e., unstable mixed layers), the relevant scaling parameters are \( w_s \) and \( z_i \). For the neutral surface layer, \( u_s \) and \( z_0 \) are applicable. Scaling parameters for surface

---

**Table 9.2** The Davenport classification, where \( z_0 \) is aerodynamic roughness length and \( C_{DN} \) is the corresponding drag coefficient for neutral static stability\(^a\)

<table>
<thead>
<tr>
<th>( z_0 ) (m)</th>
<th>Classification</th>
<th>Landscape</th>
<th>( C_{DN} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0002</td>
<td>Sea</td>
<td>Calm sea, paved areas, snow-covered flat plain, tide flat, smooth desert.</td>
<td>0.0014</td>
</tr>
<tr>
<td>0.005</td>
<td>Smooth</td>
<td>Beaches, pack ice, morass, snow-covered fields.</td>
<td>0.0028</td>
</tr>
<tr>
<td><strong>0.03</strong></td>
<td>Open</td>
<td>Grass prairie or farm fields, tundra, airports, heather.</td>
<td>0.0047</td>
</tr>
<tr>
<td>0.1</td>
<td>Roughly open</td>
<td>Cultivated area with low crops and occasional obstacles (single bushes).</td>
<td>0.0075</td>
</tr>
<tr>
<td>0.25</td>
<td>Rough</td>
<td>High crops, crops of varied height, scattered obstacles such as trees or hedgerows, vineyards.</td>
<td>0.012</td>
</tr>
<tr>
<td>0.5</td>
<td>Very rough</td>
<td>Mixed farm fields and forest clumps, orchards, scattered buildings.</td>
<td>0.018</td>
</tr>
<tr>
<td>1.0</td>
<td>Closed</td>
<td>Regular coverage with large size obstacles with open spaces roughly equal to obstacle heights, suburban houses, villages, mature forests.</td>
<td>0.030</td>
</tr>
<tr>
<td>( \geq 2 )</td>
<td>Chaotic</td>
<td>Centers of large towns and cities, irregular forests with scattered clearings.</td>
<td>0.062</td>
</tr>
</tbody>
</table>

Choose axes so \( \bar{v}_g = 0 \Rightarrow \)

\[
\bar{v} = x_5 \bar{v}_g / \bar{u} \\
\bar{u} = \bar{u}_g - x_5 \bar{v}_g / \bar{u}
\]

where

\[ x_5 = cd / (fh) . \]

wind speed \( |\bar{v}| = (\bar{u}^2 + \bar{v}_g^2)^{1/2} \) is less than \( |\bar{v}_g| = u_g \), and vector balance gives cross-isobar flow toward lower pt.

**Example**

\[
\bar{u}_g = 10 \text{ m/s} \]

\[ x_5 = 0.05 \text{ m}^{-1} \text{ s}, \]

\[ \bar{u} = 3.77 \text{ m/s} \]

\[ |\bar{v}| = 9.10 \text{ m/s}. \]

What is cross-isobar angle?

Work is done by \( p \) to balance KE loss due to friction.

**Balance**

\[
\begin{align*}
\rho & \quad \text{\( \rho - 2 \delta \rho \)} \\
\bar{v} & \quad \text{\( \rho - \delta \rho \)} \\
F & \quad \text{\( \rho \)}
\end{align*}
\]