

12) SITUATION: Daytime boundary layer over land (N. America).

OBSERVATIONS:

Height, z (m)	12	8	2	$0.1=z_0$	(m)
θ	300	301	303	308	(K)
U	5.4	5.0	3.4	0	(m/s)

What are the values of: a) C_{DN} b) C_D

- 13) For sandy clay with 15% moisture, at what depth below the surface will the diurnal temperature variations be 1% of the surface temperature variations?
- 14) Given a 1km constant thickness boundary layer with initial $\theta = 10^\circ\text{C}$ flowing at $M=10$ m/s over land, where the land has the same surface temperature as that of the air near the surface. At some point, the air leaves the land and flows over the ocean, where the ocean sea surface temperature is 20°C and the pressure is 100 kPa. Assume that the boundary layer is well mixed. Calculate and plot the heat flux Q_H and the boundary layer temperature as a function of distance from the shoreline.
- 15) For a pressure gradient of $0.2 \text{ kPa} / 100 \text{ km}$ and a surface roughness length of $z_0 = 2 \text{ cm}$, find:
- the value of the surface Rossby number
 - the value of the neutral geostrophic drag coefficient
 - u_* , assuming statically neutral conditions
- 16) Given a drag coefficient of 3×10^{-3} at $R_i = -0.4$ and $z_s = 10 \text{ m}$, how would the drag coefficient change if $z = 100 \text{ m}$?
- 17) What is the roughness length, z_0 , over the ocean for a wind speed of 40 m/s ?
- 18) What is the time lag of the diurnal cycle of temperature at a depth of 15 cm in farmland? What will be the amplitude of the temperature wave at that depth?
- 19) Use equation (7.5.2b) to calculate and plot $s_{cc}(T)$ vs T , and compare your answer with Fig 7.13.
- 20) For a temperature of 20°C , dew point of 10°C , ground surface relative humidity of $X_0 = 80\%$, and wind speed of 5 m/s , find Q_H and Q_E using:
- Priestley-Taylor method
 - Penman-Monteith method
- (Hint: use the result from the previous question.)
- 21) Verify that the surface energy balance equation is satisfied using the Q_H and Q_E parameterizations from:
- The Bowen ratio method (equations 7.5.1 b & c)
 - The Priestley-Taylor Method (equations 7.5.2e & f)
 - The Penman-Monteith method (equations 7.5.3a & b).

8 Some Mathematical & Conceptual Tools: Part 2. Time Series

Spectrum analysis is a statistical tool that we can employ to probe further into the workings of turbulence. By decomposing a series of measurements into frequency or wavenumber components, we can discover how eddies of different time and space scales contribute to the overall turbulence state.

In this chapter we review the computational techniques for the spectrum analysis of measured data. We also introduce related tools such as the autocorrelation function, structure function, and periodogram. Also discussed is the concept of a process spectrum, where mixing processes rather than turbulence states are decomposed into a spectrum of scales. Theoretical spectral decomposition of the TKE equation is briefly covered.

8.1 Time and Space Series

When measurements are taken at a fixed point over a period of time, the resulting series of data points is called a *time series*. Similarly, measurements at a fixed time over a series of locations in space is called a *space series*. Both series give measurements of a dependent variable such as temperature or humidity as a function of an independent variable, such as time, t , or location, x . Because of this similarity, we will discuss the two types of series interchangeably, and sometimes will use the generic name, *series*.

This review will be limited to *discrete* series; namely, measurements taken at regularly-spaced intervals that lead to a finite number, N , of data points. A discrete series represents a *sample* of the true, continuously-varying signal. Examples of discrete series include temperature or tracer concentration measurements made every second during the course of an hour at a fixed location such as an instrumented tower, or measurements of humidity taken every meter from an aircraft flying on a 25 km flight leg.

If $A(t)$ represents the true signal as a continuous function of time, then we could sample that signal at evenly-spaced times: $t = t_0$, $t = t_0 + \Delta t$, $t = t_0 + 2\Delta t$, $t = t_0 + 3\Delta t$, ..., $t = t_0 + (N-1)\Delta t$, where the total number of data points is N . We will use an *index*, k , to denote the position within the time series. The k^{th} data point corresponds to time $t_k = t_0 + k\Delta t$, where $0 \leq k \leq (N-1)$. Sometimes the value of variable A at time t_k is represented by $A(t_k)$, but usually the shorthand notations $A(k)$ or A_k is used. We will assume that the *sampling interval* is Δt , with no missing data and no changes of Δt within any one series. The *total period* of measurements is $P = N\Delta t$, in the sense that each of the N data points represents a sample within an interval Δt .

8.2 Autocorrelation

In section 2.4.5 we discussed the covariance and the correlation coefficient, which quantify the amount of common variation between two different variables. Extending this idea, we could also ask about the degree of common variation between a variable (A) sampled at time t and that same variable sampled at a later time, $t+L$, where L is the time lag. Such a correlation of a variable with itself is called *autocorrelation*, $R_{AA}(L)$.

Consider a 12 hour time series that has a simple sinusoidal variation of unit amplitude with a 4 hour period. The wave equals 1.0 at regular intervals of 1, 5, and 9 h. Also, the wave equals -1.0 at 3, 7, and 11 h. In fact at ANY time, t , the series is perfectly correlated with itself (i.e., has the same value) at exactly times $t + 4$ h, $t + 8$ h, and $t + 12$ h. Similarly, we can show that the wave is negatively correlated with itself at $t + 2$ h, $t + 6$ h, and $t + 10$ h. We have, in essence, just determined the autocorrelation for this series at lags 2, 4, 6, 8, 10, and 12 h.

If our time series consists of a wave that varies in frequency during the duration of the series, then a wave at t_1 might be perfectly correlated with itself at $t_1 + 4$ h, but the value at t_2 might not be perfectly correlated with the value at $t_2 + 4$ h. When averaged over all possible pairs of data points with 4 h lag in this series, the result might NOT give a large correlation value at all.

In other words, the autocorrelation measures the persistence of a wave within the whole duration a time or space series. The capability to determine persistent waves or oscillations within a series is particularly valuable because the regular variation might be associated with a physical phenomenon such as an eddy. Alternately, when the autocorrelation becomes close to zero, it tells us that there is a random process (e.g. turbulence) occurring with no persistent or regularly-recurring structures.

8.2.1 Definition

The exact definition for the discrete autocorrelation is:

$$R_{AA}(L) = \frac{\sum_{k=0}^{N-j-1} [(A_k - \bar{A}_k)(A_{k+j} - \bar{A}_{k+j})]}{\left[\sum_{k=0}^{N-j-1} (A_k - \bar{A}_k)^2 \right]^{1/2} \left[\sum_{k=0}^{N-j-1} (A_{k+j} - \bar{A}_{k+j})^2 \right]^{1/2}} \quad (8.2.1a)$$

where two different mean values are used depending on which portion of the whole series is being considered:

$$\bar{A}_k = \frac{1}{N-j} \sum_{k=0}^{N-j-1} A_k \quad \text{and} \quad \bar{A}_{k+j} = \frac{1}{N-j} \sum_{k=0}^{N-j-1} A_{k+j}$$

and where lag = $L = j \Delta t$. Notice that each of the square bracket terms in the denominator acts like a standard deviation over the portion of the data set being used.

We can approximate (8.2.1a) if it is assumed that the data is sufficiently stationary (or homogeneous for space series) that the mean values over each portion of the series is equal to the overall series mean, and that the standard deviations from each portion equal the overall series standard deviation. This results in:

$$R_{AA, \text{approx}}(L) = \frac{\overline{A_k A_{k+j}}}{\sigma_A^2} \quad (8.2.1b)$$

This simple approximation works satisfactorily for small lags (i.e., small j) and large N , but is inadequate otherwise.

Autocorrelations are usually calculated for a range of lags, and the result plotted on a graph of R_{AA} vs L . For the special case of zero lag, the autocorrelation is identically equal to unity [$R_{AA}(0) = 1.0$] for all signals. The autocorrelation of an irregular signal such as turbulence approaches zero as L approaches infinity, although it may appear as damped oscillations about zero while L is small. Also, as the lag increases, the percentage of the time series used to calculate $R_{AA}(L)$ decreases. As a result, the statistical significance of R_{AA} decreases as lag increases, making R_{AA} unrepresentative when $j > (N/2)$.

Sample autocorrelation curves for convective turbulence measured at different heights in the ML are shown in Fig 8.1 (Deardorff and Willis, 1985).

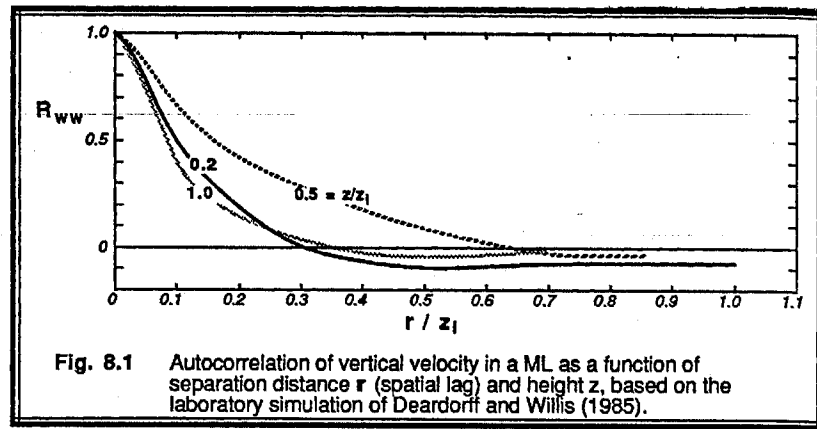


Fig. 8.1 Autocorrelation of vertical velocity in a ML as a function of separation distance r (spatial lag) and height z , based on the laboratory simulation of Deardorff and Willis (1985).

8.2.2 Example

Problem: Given the following series of measurements of relative humidity made every 3 h over a 96 h period (4 days) at a fixed point. Find the autocorrelation for relative humidity, $R_{rh,rh}(L)$, for time lags ranging from 0 to 48 hours, and plot the result.

Data:

	Relative humidity (percent)							
Day 1:	49	46	44	45	52	59	61	57
Day 2:	53	50	50	52	55	55	54	47
Day 3:	41	36	32	33	36	41	40	37
Day 4:	34	31	29	32	38	45	48	45

As can be seen in a plot of the time series (Fig 8.2a), there are regular diurnal cycle oscillations superimposed on longer period trends.

Solution: There are 32 data points, with $\Delta t = 3$ h. We must solve (8.2.1) 17 different times, for $j = 0$ through $j = 16$. The result is:

L(h)	$R_{rh,rh}$	L(h)	$R_{rh,rh}$	L(h)	$R_{rh,rh}$	L(h)	$R_{rh,rh}$	L(h)	$R_{rh,rh}$	L(h)	$R_{rh,rh}$
0	1.00	9	0.35	18	0.49	27	0.34	36	-0.13	45	0.40
3	0.67	12	0.34	21	0.53	30	0.09	39	-0.04	48	0.58
6	0.47	15	0.40	24	0.50	33	-0.12	42	0.17		

Discussion: Looking at Fig 8.2b, we see that the autocorrelation starts at 1.0 at zero lag, and quickly decreases. As is sometimes the case with weather data, the diurnal cycle shows up as an oscillation in the autocorrelation function with a 24 hour period. We could have anticipated this, because 12 h from any time, the humidity time series is in the

opposite side of its oscillation. If the humidity is high in the early morning, then 12 h later it is drier. If the humidity is low in the afternoon, then 12 h later it is more humid. On the average, humidity is negatively correlated with itself 12 h later. Over a 24 h period, however, like comparing a morning humidity with the next morning's humidity, or the afternoon humidity with the next afternoon's humidity, we anticipate a positive correlation.

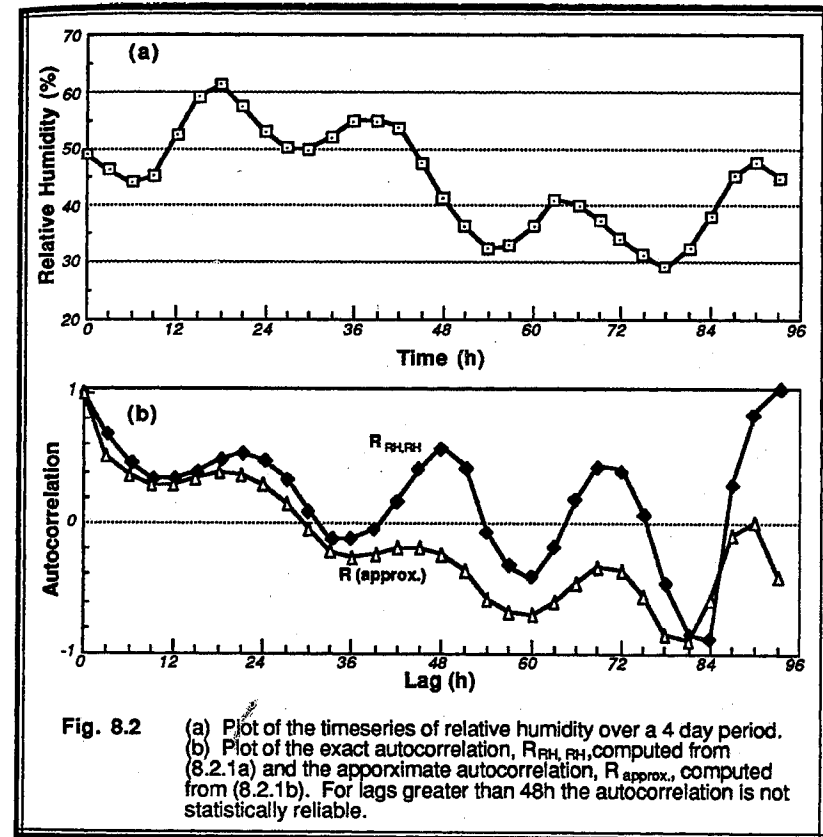


Fig. 8.2 (a) Plot of the timeseries of relative humidity over a 4 day period. (b) Plot of the exact autocorrelation, $R_{RH,RH}$, computed from (8.2.1a) and the approximate autocorrelation, R_{approx} , computed from (8.2.1b). For lags greater than 48h the autocorrelation is not statistically reliable.

The initial drop off of the autocorrelation from 1.0 to smaller values is a measure of the accuracy of a persistence forecast. Namely, if we forecast the humidity 3 or less hours from now to be the same as the present humidity, we would probably be close to correct because the autocorrelation is 60% or higher. Longer forecasts would be less accurate.

8.3.2 Example

Problem: Given the same time series of relative humidity (rh) as in example 8.2.2, calculate the structure function for lags from 0 - 48 hours.

Solution: Using (8.3.1a), the structure function values (in units of relative humidity percentage squared) are:

L (h)	D _{rh,rh}	L(h)	D _{rh,rh}	L(h)	D _{rh,rh}	L(h)	D _{rh,rh}
0	0	15	153	27	183	39	301
3	78	18	134	30	243	42	303
6	124	21	127	33	299	45	326
9	155	24	142	36	316	48	377
12	162						

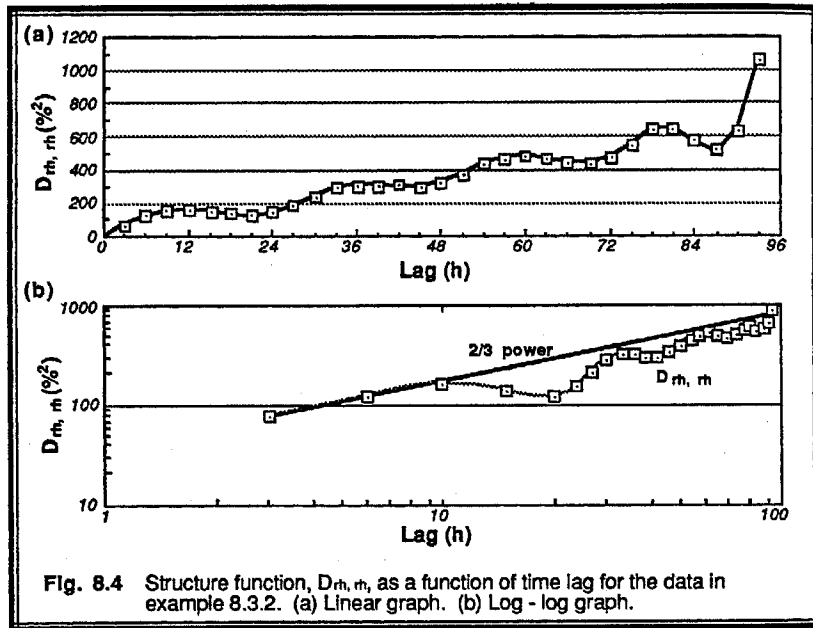


Fig. 8.4 Structure function, $D_{rh,rh}$, as a function of time lag for the data in example 8.3.2. (a) Linear graph. (b) Log - log graph.

Discussion: Fig 8.4a shows the resulting variation of the structure function on a linear scale. A similar plot on log-log graph is given in Fig 8.4b. The straight line on this latter graph is given by $D_{rh,rh} = c_{rh^2} L^{2/3}$, with the structure function for relative humidity $c_{rh^2} = 35 (\% ^2 \cdot h^{-2/3})$.

8.4 Discrete Fourier Transform

From Fourier analysis in calculus we remember that any well-behaved continuous function can be described by an infinite Fourier series — namely, the sum of an infinite number of sine and cosine terms. In the case of a discrete time series with a finite number of points, we are required to have only a finite number of sine and cosine terms to fit our points exactly.

8.4.1 Definition

Using Euler's notation [$\exp(ix) = \cos(x) + i \sin(x)$, where i is the square root of -1] as a shorthand notation for the sines and cosines, we can write the discrete Fourier series representation of $A(k)$ as:

$$\text{Inverse Transform: } A(k) = \sum_{n=0}^{N-1} F_A(n) e^{i2\pi nk/N} \quad (8.4.1a)$$

where n is the frequency, and $F_A(n)$ is the *discrete Fourier transform*. We see that a time series with N data points (indexed from $k=0$ through $N-1$) needs no more than N different frequencies to describe it (actually, it needs less than N , as will be shown later).

There are a number of ways to describe frequency:

- n = number of cycles (per time period P),
- \bar{n} = cycles per second = n/P ,
- f = radians per second = $2\pi n/P = 2\pi \bar{n}/(N\Delta t)$.

A frequency of zero ($n = 0$) denotes a mean value. The *fundamental frequency*, where $n = 1$, means that exactly one wave fills the whole time period, P . Higher frequencies correspond to *harmonics* of the fundamental frequency. For example, $n = 5$ means that exactly 5 waves fill the period P .

$F_A(n)$ is a complex number, where the real part represents the amplitude of the cosine waves and the imaginary part is the sine wave amplitude. It is a function of frequency because the waves of different frequencies must be multiplied by different amplitudes to reconstruct the original time series. If the original time series $A(k)$ is known, then these coefficients can be found from:

$$\text{Forward Transform: } F_A(n) = \sum_{k=0}^{N-1} \left[\frac{A(k)}{N} \right] e^{-i2\pi nk/N} \quad (8.4.1b)$$

Notice the similarity between (8.4.1a) and (8.4.1b). These two equations are called *Fourier transform pairs*. The second equation performs the *forward transform*, creating a representation of the signal in *phase space* (another name for the frequency

or spectral domain). This process is also known as *Fourier decomposition*. The first equation performs the *inverse transform*, converting from frequencies back into *physical space*.

8.4.2 Example

Problem: Given the following 8 data points of specific humidity, q , as a function of time:

Index (k):	0	1	2	3	4	5	6	7
Time (UTC):	1200	1215	1230	1245	1300	1315	1330	1345
q (g/kg):	8	9	9	6	10	3	5	6

Perform a forward Fourier transform to find the 8 coefficients, $F_q(n)$. To check your results, perform an inverse transform to confirm that the original time series is recreated. Remember that the $F_q(n)$ coefficients are complex, each having a real and an imaginary part: $F_q(n) = F_{real}(n) + iF_{imag}(n)$.

Solution: $N = 8$ and $\Delta t = 15$ min. Thus, the total period is $P = N\Delta t = 2$ h. Equation (8.4.1b) must be used to find $F_q(n)$. For those computer languages that accept complex numbers, (8.4.1b) can be programmed directly, where each of the $A(k)$ data points has a real part equal to the value listed in the table, and an imaginary part of zero.

For hand calculation, we can use Euler's formula to translate (8.4.1b) back into sines and cosines:

$$F_A(n) = \frac{1}{N} \sum_{k=0}^{N-1} A(k) \cos(2\pi nk/N) - \frac{i}{N} \sum_{k=0}^{N-1} A(k) \sin(2\pi nk/N)$$

As an example, for $n = 0$, all of the cosines of zero are unity and all of the sines are zero. This leaves:

$$F_A(0) = \frac{1}{N} \sum_{k=0}^{N-1} A(k)$$

which is just the mean of A . For our case: $F_q(0) = 7.0 - 0.0i$. For $n = 1$ we can't make such a simplification, so we are forced to sum over all k for both the real and imaginary parts. This gives us $F_q(1) = 0.28 - 1.03i$. Continuing this procedure for all other n yields:

n	$F_q(n)$	n	$F_q(n)$
0	7.0	4	1.0
1	0.28 - 1.03 i	5	-0.78 + 0.03 i
2	0.5	6	0.5
3	-0.78 - 0.03 i	7	0.28 + 1.03 i

This is the answer to the first part of the problem. Note that for frequencies greater than 4, the Fourier transform is just the complex conjugate of the frequencies less than 4.

As a check of our transform, we can perform the inverse transform using (8.4.1a) directly in a computer program. Otherwise, we can use Euler's formula to write it as:

$$A(k) = \sum_{n=0}^{N-1} F(n)_{(real\ part)} \cdot \cos(2\pi nk/N) - \sum_{n=0}^{N-1} F(n)_{(imag.\ part)} \cdot \sin(2\pi nk/N)$$

In actuality, there are four sums, not just the two listed above. The remaining sums consist of the real part of F times the imaginary factor $i \cdot \sin(\dots)$, and the imaginary part of F times the real factor $\cos(\dots)$. Because the last half of the Fourier transforms are the complex conjugates of the first half (not counting the mean), these two sums identically cancel, leaving the two listed above. Upon performing the calculations for $A(k)$, we do indeed reproduce the original time series.

Discussion: To graphically demonstrate that the sum of these sines and cosines does indeed equal our original series, Fig 8.5 shows each individual wave multiplied by its appropriate amplitude. As can be seen, the reconstructed time series fits perfectly the eight original data points. In between these points, however, the sum oscillates in a manner that is not necessarily realistic, but which is irrelevant because it occurs below the discretization resolution specified by the original data points.

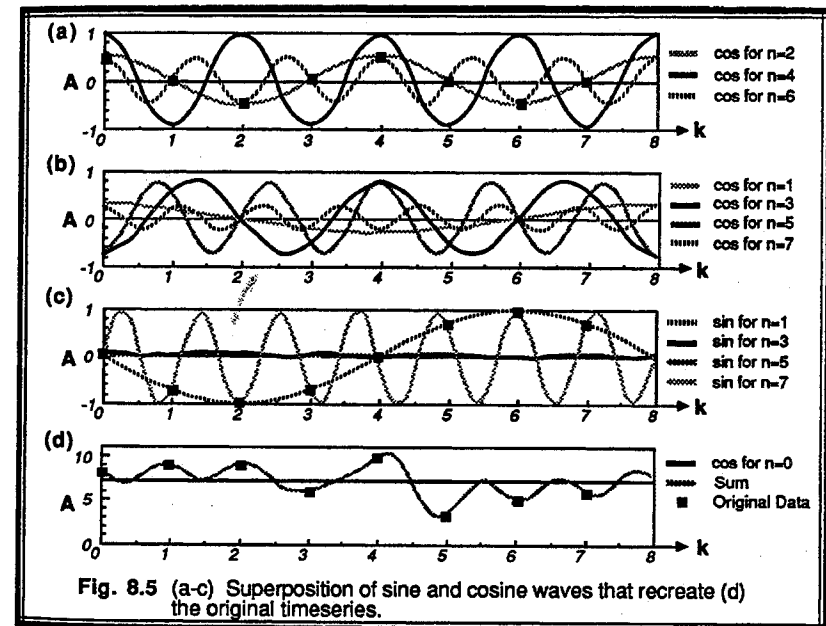


Fig. 8.5 (a-c) Superposition of sine and cosine waves that recreate (d) the original timeseries.

have "canned" FFT algorithms that users can access without having to write their own. Some of the early FFT packages were restricted to data sets with $N = 2^m$, where m was any integer. This meant that data sets slightly too long were truncated to the proper size, or data sets slightly too short were lengthened by adding bogus data (often zeros or the mean value). Both of these data mutilation tricks are not recommended. Modern FFTs factor the series into a variety of prime numbers in addition to the prime number 2, resulting in very little truncation of the time series.

One problem with all discrete Fourier transforms including FFTs, is that the input must consist of equally-spaced data points. No missing data is allowed. If the data set has gaps caused by instrument failures or by spurious data spikes that were removed, then artificial data points must be inserted to fill the gap. One is not allowed simply to close the gap by bringing the remaining parts of the data set together, because this alters the periods or wavelengths present in the original signal. The artificial data points must be chosen with care, otherwise this "fudge" can destroy an otherwise unbiased data set. Data with significant gaps can be analyzed with periodogram methods instead (see Section 8.9).

8.6 Energy Spectrum

8.6.1 Discrete Energy Spectrum

In meteorology we are frequently curious about how much of the variance of a time series is associated with a particular frequency, without regard to the precise phase of the waves. Indeed for turbulence, we anticipate that the original signal is not physically like waves at all, but we still find it useful to break the signal into components of different frequencies that we like to associate with different eddy sizes.

The square of the norm of the complex Fourier transform for any frequency n is:

$$|F_A(n)|^2 = [F_{\text{real part}(n)}]^2 + [F_{\text{imag. part}(n)}]^2 \quad (8.6.1a)$$

When $|F_A(n)|^2$ is summed over frequencies $n = 1$ to $N-1$, the result equals the total biased variance of the original time series:

$$\sigma_A^2 = \frac{1}{N} \sum_{k=0}^{N-1} (A_k - \bar{A})^2 = \sum_{n=1}^{N-1} |F_A(n)|^2 \quad (8.6.1b)$$

Thus, we can interpret $|F_A(n)|^2$ as the portion of variance explained by waves of frequency n . Notice that the sum over frequencies does not include $n=0$, because $|F_A(0)|$ is the mean value and does not contribute any information about the variation of the signal about the mean. To simplify the notation for later use, define: $G_A(n) = |F_A(n)|^2$. The ratio $G_A(n) / \sigma_A^2$ represents the fraction of variance explained by component n , and is

very much like the correlation coefficient squared, r^2 .

For frequencies greater than the Nyquist frequency the $|F_A(n)|^2$ values are identically equal to those at the corresponding folded lower frequencies, because the Fourier transforms of high frequencies are the same as those for the low frequencies, except for a sign change in front of the imaginary part. Also, since frequencies higher than the Nyquist cannot be resolved anyway, the $|F_A(n)|^2$ values at high frequencies should be folded back and added to those at the lower frequencies.

Thus, *discrete spectral intensity (or energy)*, $E_A(n)$, is defined as $E_A(n) = 2 \cdot |F_A(n)|^2$, for $n = 1$ to n_f , with $N = \text{odd}$. For $N = \text{even}$, $E_A(n) = 2 \cdot |F_A(n)|^2$ is used for frequencies from $n = 1$ to $(n_f - 1)$, along with $E_A(n) = |F_A(n)|^2$ (not times 2) at the Nyquist frequency. This presentation is called the *discrete variance (or energy) spectrum*. It can be used for any variable such as temperature, velocity, or humidity to separate the total variance into the components, $E_A(n)$, related to different frequencies. For variables such as temperature and humidity, however, we must not associate the resulting spectrum with concepts of eddy motions, because variations in these variables can persist in the atmosphere in nonturbulent flow as the "footprints" of formerly active turbulence.

The variance of velocity fluctuations, u' , has the same units as turbulence kinetic energy per unit mass. Thus, the spectrum of velocity is called the *discrete energy spectrum*. As defined above, the name "energy spectrum" is sometimes used for all variance spectra.

8.6.2 Spectral Density

Although this chapter has dealt with discrete spectra, a number of theoretical concepts such as the spectral similarity discussed in the next chapter use continuous spectral representations. Namely, instead of summing the discrete spectral energy over all n to yield the total variance, these theories assume that there is a *spectral energy density*, $S_A(n)$ that can be integrated over n to yield the total variance.

$$\sigma_A^2 = \int_n S_A(n) dn \quad (8.6.2a)$$

The spectral energy density has units of A squared per unit frequency.

We can approximate the spectral energy density by

$$S_A(n) = \frac{E_A(n)}{\Delta n} \quad (8.6.2b)$$

where Δn is the difference between neighboring frequencies. When n is used to represent frequency, $\Delta n = 1$. For other representations of frequency such as f , we will find that Δf is not necessarily equal to unity.

The $S_A(n)$ points estimated from (8.6.2b) can then be connected with a smooth curve to represent the shape of the spectrum. An example of this was shown in Chapter 2, Fig. 2.2. Thus, even with discrete meteorological data, we can estimate spectral densities that can be compared to theories.

8.6.3 Example

Problem: Use the results from the $N = 8$ data point example of section 8.4.2 to calculate the discrete spectral energies for all frequencies. Plot the result in the usual presentation format for discrete spectra. Show an additional graph of the estimate of spectral density.

Solution:

n	$F_q(n)$	$ F_q(n) ^2$	$E_q(n)$	$S_q(n)$
0	7.0 (= mean)			
1	0.28 - 1.03 i	1.14	2.28	2.28
2	0.5	0.25	0.5	0.5
3	-0.78 - 0.03 i	0.61	1.22	1.22
4 = n_f	1.0	1.0	1.0	1.0
5	-0.78 + 0.03 i	0.61		
6	0.5	0.25		
7	0.28 + 1.03 i	1.14		
	Sum =	5.0	=	5.0

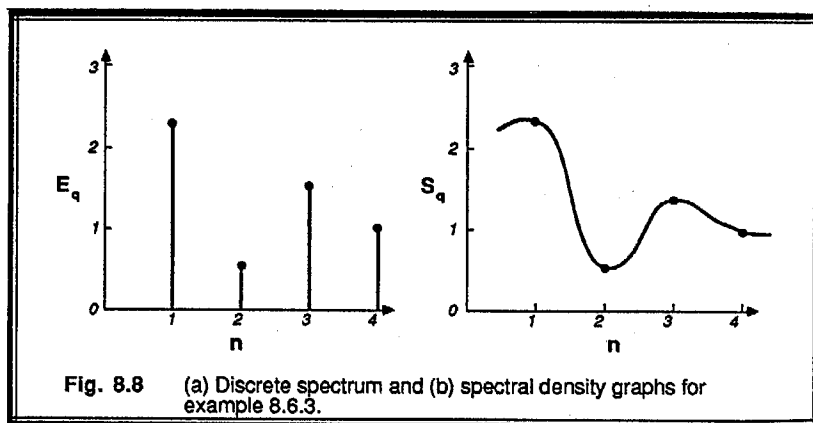


Fig. 8.8 (a) Discrete spectrum and (b) spectral density graphs for example 8.6.3.

where $E_q(n)$ has units of specific humidity squared, and $S_q(n)$ has units of specific humidity squared per unit frequency. Finally, the discrete spectrum is plotted in Fig 8.8a, and the spectral energy density is plotted in Fig 8.8b.

Discussion. The sum of the spectral energies equals the biased variance of the original signal, $\sigma_q^2 = 5.0$. This is always a good check of the FFT for you to perform.

8.6.4 Graphical Presentation of Atmospheric Spectra

A wide range of intensities are present in atmospheric turbulence spectra over an even larger range of frequencies. Atmospheric turbulence spectral energies characteristically peak at the lowest frequencies, namely at about 1 to 10 cycles per hour. At higher frequencies, the spectral energy decreases. For example, at frequencies of 10^4 cycles per hour the energy is one to two orders of magnitude smaller than at the peak.

We are often concerned about the full range of the spectrum: the peak is associated with the production of turbulence and usually the largest eddy sizes; the middle frequencies are associated with the inertial subrange, which is important for estimated dissipation rates; and the highest frequencies are associated with the dissipation of TKE into heat by viscous effects. Hence, we need a way to graphically present the spectral data in a form that not only highlights the important peaks and other characteristics, but which shows all portions of the wide range of data.

In the discussions that follow, a single idealized spectrum is presented in a variety of formats in Fig 8.9. The data for these plots is listed in Table 8-1.

Linear-linear presentation. When $S_A(f)$ is plotted vs. f on a linear-linear graph, the result has the desirable characteristic that the area under the curve between any pair of frequencies is proportional to the portion of variance explained by that range of frequencies. Unfortunately, the plot is useless to view because the wide range in values results in a compression of the data onto the coordinate axes (see Fig 8.9a). Alternatives include expanding the low frequency portion of the spectrum (Fig 8.9b) and plotting $f \cdot S(f)$ instead of just $S(f)$ on the ordinate (Fig 8.9c). Both techniques focus on the spectral peak at the expense of losing information at the higher frequencies.

Note that the $f \cdot S(f)$ plot causes the apparent peak to shift from the low frequency end of the spectrum towards the middle of the spectrum. Since $f \cdot S(f)$ is also used in a number of the other formats listed below, we should not be deceived into thinking that the middle frequencies are the ones with the most spectral energy.

Semi-log presentation. By plotting $f \cdot S_A(f)$ vs. $\log f$, the low frequency portions of the spectra are expanded along the abscissa. Also, the ordinate for the high frequency portions are enhanced because the spectral density is multiplied by frequency (see Fig 8.9d). Another excellent quality is that the area under any portion of the curve continues to be proportional to the variance.

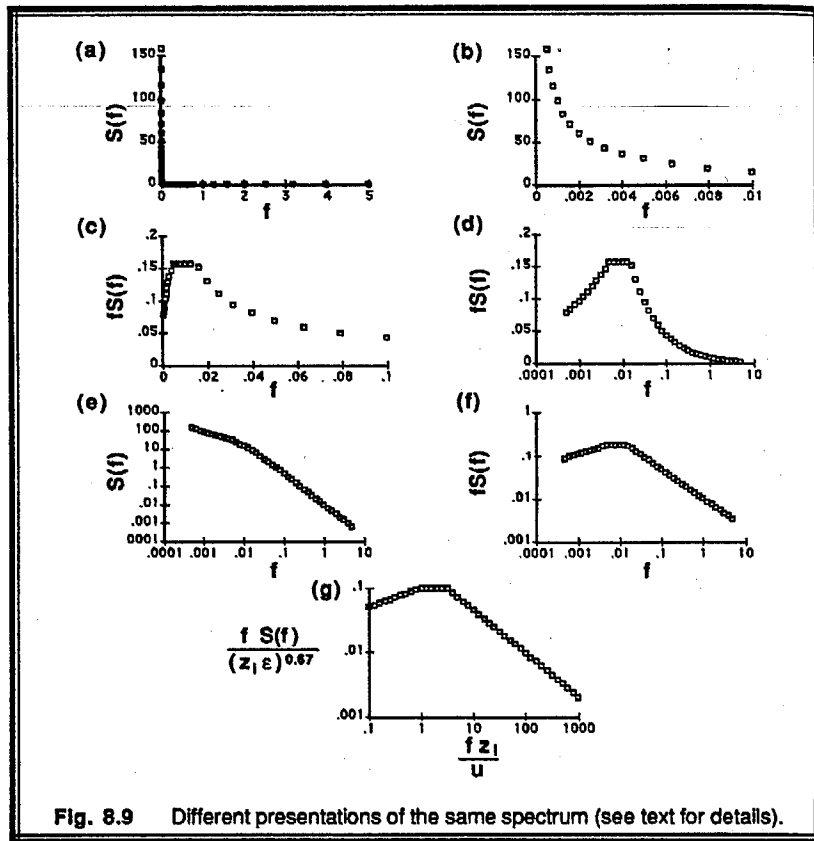


Fig. 8.9 Different presentations of the same spectrum (see text for details).

Log-log presentation. When $\log[S_A(f)]$ vs. $\log f$ is plotted, the result allows a wide range of frequencies and spectral densities to be displayed. Also, any power law relationships between $S_A(f)$ and f appear as straight lines on this graph. As will be discussed in more detail in the next chapter, $S_A(f)$ is proportional to $f^{-5/3}$ in the inertial subrange portion of the spectrum, which will appear as a straight line with $-5/3$ slope on a log-log graph (see Fig. 8.9e). Unfortunately, the area under the curve is no longer proportional to the variance.

Log $f S_A(f)$ vs. $\log f$. A plot of $\log[f \cdot S_A(f)]$ vs. $\log f$, has all of the desirable characteristics of the log-log presentation described above. In addition, the quantity $f \cdot S_A(f)$ has the same units as the variance of A , making scaling or normalization easier. Unfortunately, the area under the curve is also not proportional to variance (see Fig. 8.9f). Regardless of this problem, this presentation is the most used in the literature.

Table 8-1. Artificial data and spreadsheet calculations used to demonstrate various ways to present spectra.

				Variable	Value
This is assumed to be the spectrum of a time series of velocity measurements.				Zi (m)	1000
				U (m/s)	5
				Dissip.(m ² s ⁻³)	0.00?
				Size	21

Logarithm of		Normalized	Normalized	f	S	fS
Normalized	Normalized	Normalized	Normalized	(1/s)	(m ² /s ³)	(m ² /s ²)
Frequency	Spectrum	Frequency	Spectrum			
-1.0	-1.3010	0.1000	0.0500	0.0005	158.7401	0.0794
-0.8	-1.2412	0.1580	0.0574	0.0008	115.3005	0.0911
-0.6	-1.1807	0.2510	0.0660	0.0013	83.4309	0.1047
-0.4	-1.1204	0.3980	0.0758	0.0020	60.4486	0.1203
-0.2	-1.0602	0.6310	0.0871	0.0032	43.8016	0.1382
-0.0	-1.0000	1.0000	0.1000	0.0050	31.7480	0.1587
0.2	-1.0000	1.5850	0.1000	0.0079	20.0303	0.1587
0.4	-1.0000	2.5120	0.1000	0.0126	12.6385	0.1587
0.6	-1.0827	3.9810	0.0827	0.0199	6.5914	0.1312
0.8	-1.2175	6.3100	0.0606	0.0316	3.0495	0.0962
1.0	-1.3521	10.0000	0.0445	0.0500	1.4112	0.0706
1.2	-1.4868	15.8490	0.0326	0.0792	0.6530	0.0517
1.4	-1.6215	25.1190	0.0239	0.1256	0.3022	0.0379
1.6	-1.7562	39.8110	0.0175	0.1991	0.1398	0.0278
1.8	-1.8909	63.0960	0.0129	0.3155	0.0647	0.0204
2.0	-2.0255	100.0000	0.0094	0.5000	0.0299	0.0150
2.2	-2.1602	158.4890	0.0069	0.7924	0.0139	0.0110
2.4	-2.2949	251.1890	0.0051	1.2559	0.0064	0.0080
2.6	-2.4296	398.1070	0.0037	1.9905	0.0030	0.0059
2.8	-2.5643	630.9570	0.0027	3.1548	0.0014	0.0043
3.0	-2.6990	1000.0000	0.0020	5.0000	0.0006	0.0032

As will be discussed in the next chapter, both the abscissa and ordinate are often made dimensionless by normalizing with respect to scaling variables (see Fig 8.9g). The scaling variables used in this example are listed in Table 8-1.

8.7 Spectral Characteristics

Instead of discussing spectral behavior theoretically, this section demonstrates spectral behavior for a single variable through a series of examples with synthetic data. In each of the following cases, an artificial time series of 20 data points is plotted, along with the spectrum computed with an FFT program. The spectrum shows $E(n)$ normalized by the total biased variance, and shows the fraction of the total variance explained by each frequency. The Nyquist frequency is $n=10$ for all cases.

Case A (Fig 8.10a): Simple waves of one frequency. All of these examples show a wave having four cycles per time period. The first four examples in this case show that the spectrum is independent of the phase of the original time series. A single simple wave in physical space produces a single spike in the spectrum at $n=4$ that explains all the variance. The fifth example shows that if the spectrum is normalized by the total variance, we still have a single spike that explains 100% of the variance. If the spectrum had not been normalized, the spike for this fifth case would have been twice as large as the spikes for the other four cases, because the time series for the fifth case consisted of a wave with twice the amplitude.

Case B (Fig 8.10b): Simple waves of different frequencies. The first example shows a time series filled by one wave, resulting in a spectrum with a spike at $n = 1$. The next three examples show waves with 4, 8, and 10 cycles per period in the time series, resulting in spectra with frequency spikes at $n = 4, 8,$ and 10 respectively. The fifth example shows a time series with a wave having 12 cycles per period, but the aliasing problem causes this signal to be folded back to $n = 8$, where it appears as a spike on the spectrum.

Case C (Fig 8.10c): Frequencies between resolvable frequencies. The FFT consists of waves of the fundamental frequency ($n = 1$) and only the exact harmonics ($n = 2, 3, 4, \dots$). But what happens if the real signal has a frequency of $n = 4.2$ or 4.5 ? These examples show that a wave of $n = 4.5$ appears as two spikes at $n = 4$ and $n = 5$. The closer the signal is to an exact harmonic, the greater the spectral energy at that harmonic and the smaller the energy at the next nearest neighbor. Notice that for a signal with $n = 4.5$, the spectrum not only has the two large spikes described above, but there is also a leakage of some small amount of spectral energy to all the other frequencies. We might expect that a real turbulence signal consisting of a multitude of frequencies, many of which are not exact harmonics of the fundamental frequency, will result in a spectrum with a lot of leakage, making it difficult to separate the true signals from the underlying noise.

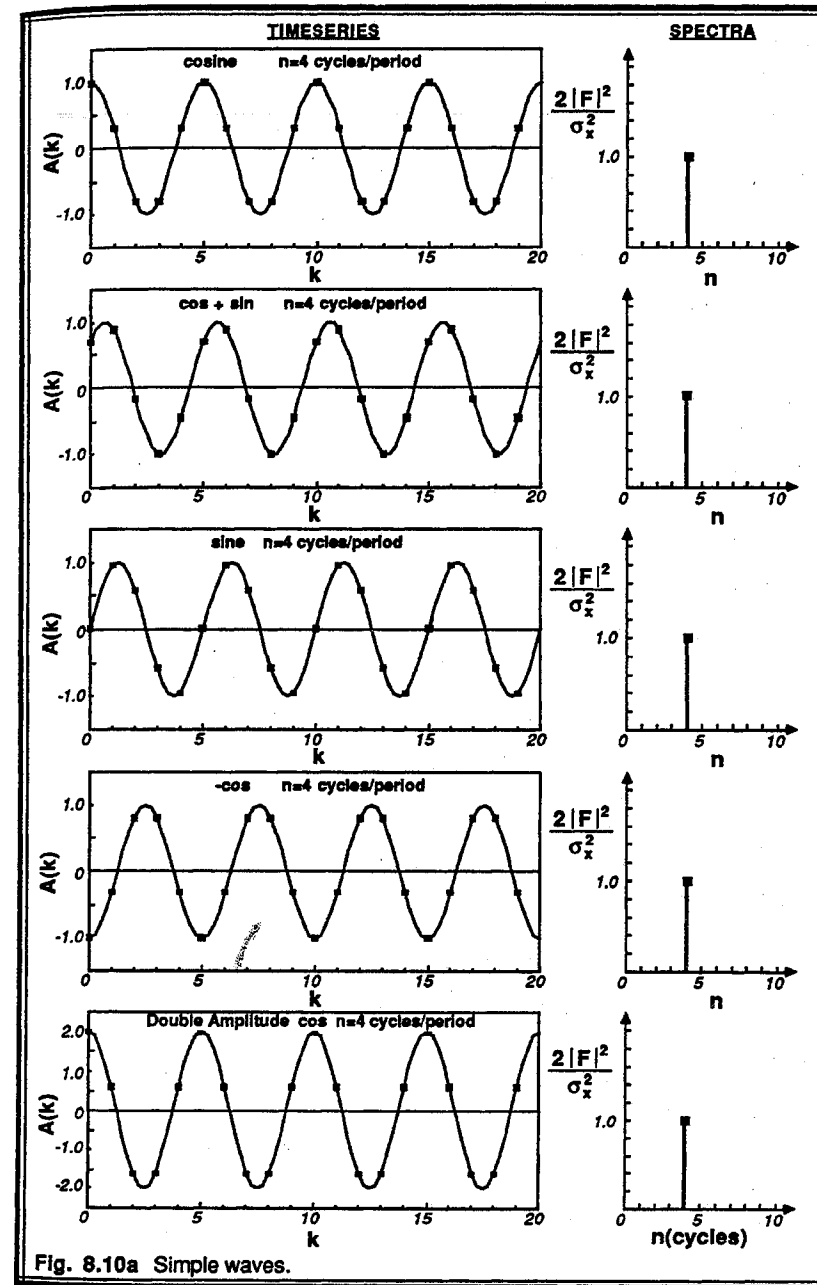


Fig. 8.10a Simple waves.