Meteorology 6160
Conserved Variables

Dry static energy: $s = c_p T + gz$

Show that s is conserved (that is, $ds = 0$) for dry (unsaturated) adiabatic, hydrostatic processes. Use the first law of thermodynamics for dry adiabatic processes,

$$0 = c_p\, dT - \alpha dp, \tag{1}$$

where $\alpha = 1/\rho$, and the hydrostatic equation,

$$\frac{dp}{dz} = -\rho g,$$

which can be written as

$$\alpha\, dp = -g\, dz. \tag{2}$$

Use (2) in (1):

$$0 = c_p\, dT + g\, dz = ds. \tag{3}$$

From (3) we obtain the dry adiatic lapse rate:

$$\Gamma_d \equiv -\frac{dT}{dz} = \frac{g}{c_p}.$$

Moist static energy: $h = s + Lq = c_p T + Lq + gz$

Show that h is conserved for dry and saturated adiabatic, hydrostatic processes. For dry adiabatic processes, the water vapor mixing ratio, q, is conserved, so $dq = 0$, and therefore, $dh = ds + Ldq = 0$, and h is also conserved.

For saturated adiabatic processes, $q = q_s(T, p)$, where q_s is the saturation mixing ratio, and the first law of thermodynamics is

$$-L\, dq_s = c_p\, dT - \alpha dp = c_p\, dT + g\, dz.$$

Therefore,

$$0 = c_p\, dT + g\, dz + L\, dq_s = dh. \tag{4}$$

From (4) we obtain the saturated adiabatic lapse rate:

$$\Gamma_s \equiv -\frac{dT}{dz} = \frac{g}{c_p} + \frac{L}{c_p}\frac{dq_s}{dz} < \frac{g}{c_p}.$$

Total water mixing ratio: $q_t = q + l$

The total water mixing ratio, q_t, is the sum of the water vapor, q, and liquid water, l, mixing ratios. It is conserved for dry and saturated adiabatic processes, but not for pseudo-adiabatic processes, in which condensate falls out immediately.

Liquid water static energy: $s_l = h - Lq_t = s - Ll = c_p T - Ll + gz$

Liquid water static energy is conserved because it is a linear combination of two conserved quantities, h and q_t.

Liquid water potential temperature: $\theta_l = \theta - (L/c_p)\, l$

For shallow layers,

$$\theta_e \approx \theta + \frac{L}{c_p} q_s,$$

where θ_e is the equivalent potential temperature, which is nearly conserved for saturated adiabatic processes. Therefore, we can linearly combine θ_e and q_t to form

$$\theta_l = \theta_e - Lq_t = \theta - \frac{L}{c_p} l.$$