Meteorology 6150
Cloud System Modeling

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1 Fundamental Equations

1.1 The Basic Equations

1.1.1 Equation of motion

The movement of air in the atmosphere is governed by *Newton's Second Law*:

$$\frac{D\mathbf{v}}{Dt} = -2\mathbf{\Omega} \times \mathbf{v} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}, \tag{1.1}$$

where t is time, Ω is the angular velocity of the earth's rotation, $\mathbf{v}$ is the air velocity, ρ is density, p is pressure, ∇ is the gradient operator, and D/Dt is the total derivative:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla.$$

The terms on the right-hand side of (1.1) are the forces that affect the motion of air: Coriolis, pressure gradient, gravity (centrifugal force plus gravitation), and frictional. Note that $\mathbf{g} = -g\mathbf{k}$.

1.1.2 Equation of state

The *equation of state* for an ideal gas is

$$p = \rho R_d T, \tag{1.2}$$

where R_d is the gas constant for dry air and T is the (absolute) temperature. For a mixture of dry air and water vapor, (1.2) becomes

$$p = \rho R_d T_v, \tag{1.3}$$

where T_v is the *virtual temperature*:

$$T_v \approx T(1 + 0.61 q_v), \tag{1.4}$$

and q_v is the mixing ratio of water vapor.

1.1.3 Thermodynamic equation

The *First Law of Thermodynamics* describes the conservation of energy:

$$\dot{H} = c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt}, \tag{1.5}$$

where $\dot{H}$ is the heating rate, c_v is the specific heat capacity of dry air at constant volume, and $\alpha = 1/\rho$ is the specific volume. Use (1.2) to write (1.5) as

$$\dot{H} = c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt}, \tag{1.6}$$

where c_p is the specific heat capacity of dry air at constant pressure.

It is often more useful to write the First Law in terms of *potential temperature*,

$$\theta \equiv \frac{T}{\Pi},$$

where Π, called the *Exner function*, is defined as

$$\Pi \equiv \left(\frac{p}{p_r} \right)^{R_d/c_p}, \tag{1.7}$$

and $p_r = 1000$ hPa. In terms of θ, (1.6) becomes

$$\frac{D\theta}{Dt} = \frac{\dot{H}}{c_p \Pi}. \tag{1.8}$$

In cloud dynamics, $\dot{H}$ includes heating and cooling due to phase changes of water, absorption or emission of radiation, and molecular diffusion. In the absence of heating or cooling, (1.8) reduces to the adiabatic form,

$$\frac{D\theta}{Dt} = 0.$$

For cloud dynamics, this form is generally not useful. When the heating or cooling is due only to condensation or evaporation, $\dot{H} = -L\,Dq_v/Dt$, where L is the latent heat of vaporization, and (1.8) becomes

$$\frac{D\theta}{Dt} = -\frac{L}{c_p\Pi}\frac{Dq_v}{Dt}. \tag{1.9}$$

1.1.4 Mass conservation equation

The *continuity equation* describes the conservation of air mass:

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{v}. \tag{1.10}$$

1.1.5 Water conservation equations

Conservation of the mass of water per unit mass of air is described by a set of equations:

$$\frac{Dq_i}{Dt} = S_i,\ \text{for } i = 1,\ldots,n, \tag{1.11}$$

where q_i is the mixing ratio of the ith water species and S_i is the net source for that species.

1.2 Scale Analysis

1.2.1 Equation of motion and buoyancy

If we decompose p and ρ into a hydrostatic basic state, $p_0(z)$ and $\rho_0(z)$, and deviations from this state, $p^{'}$ and $\rho^{'}$, and apply scale analysis based on observations that indicate that $p^{'}/p_0 \ll 1$ and $\rho^{'}/\rho_0 \ll 1$, we obtain

$$-\frac{1}{\rho}\nabla p + \mathbf{g} \approx -\frac{1}{\rho_0}\nabla p^{'} + \mathbf{g}\frac{\rho^{'}}{\rho_0}, \tag{1.12}$$

where $\mathbf{g}\rho^{'}/\rho_0$ is the *buoyancy* acceleration. In addition, for cloud-scale motions, the Coriolis and frictional accelerations can be neglected. Then (1.1) becomes

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho_0}\nabla p^{'} + \mathbf{g}\frac{\rho^{'}}{\rho_0}. \tag{1.13}$$

Based on observations, $p^{'}/p_0 \ll 1$, $\rho^{'}/\rho_0 \ll 1$, and $T^{'}/T_0 \ll 1$, so the equation of state for dry air (1.2) for the deviations can be written as

$$\frac{\rho^{'}}{\rho_0} \approx \frac{p^{'}}{p_0} - \frac{T^{'}}{T_0}, \tag{1.14}$$

or, using potential temperature instead of temperature, as

$$\frac{\rho^{'}}{\rho_0} \approx \frac{c_v}{c_p}\frac{p^{'}}{p_0} - \frac{\theta^{'}}{\theta_0}. \tag{1.15}$$

Eqs. (1.14) and (1.15) allow the buoyancy to be expressed in terms of pressure and temperature or potential temperature deviations. It is shown in section 1.2.4) that the contributions of pressure deviations to buoyancy may be neglected if

$$\frac{U^2}{c^2} \ll \frac{T^{'}}{T_0},$$

where U is a typical velocity variation and

$$c \equiv (\frac{c_v}{c_p}R_d T)^{1/2} \tag{1.16}$$

is the speed of sound. Then (1.14) and (1.15) become

$$\frac{\rho^{'}}{\rho_0} \approx -\frac{T^{'}}{T_0}, \tag{1.17}$$

and

$$\frac{\rho^{'}}{\rho_0} \approx -\frac{\theta^{'}}{\theta_0}. \tag{1.18}$$

One may use the Exner function, (1.7), to write

$$-\frac{1}{\rho}\nabla p = -c_p\theta\nabla\pi.$$

If we decompose π and θ into a hydrostatic basic state, $\pi_0(z)$ and $\theta_0(z)$, and deviations from this state, $\pi^{'}$ and $\theta^{'}$, and apply scale analysis, we obtain

$$-c_p\theta\nabla\pi - \mathbf{g} \approx -c_p\theta_0\nabla\pi^{'} - \mathbf{g}\frac{\theta^{'}}{\theta_0}, \tag{1.19}$$

where $-\mathbf{g}\theta^{'}/\theta_0$ is the buoyancy acceleration. In this form, the pressure deviation does not appear in the buoyancy acceleration. For cloud-scale motions, (1.1) becomes

$$\frac{D\mathbf{v}}{Dt} = -c_p\theta_0\nabla\pi^{'} - \mathbf{g}\frac{\theta^{'}}{\theta_0}. \tag{1.20}$$

1.2.2 Compressible equations

The following prognostic equation for the non-dimensional pressure deviation, $\pi^{'}$, can be derived:

$$\frac{\partial\pi^{'}}{\partial t} = -\frac{c_o^2}{c_p\rho_0\theta_o^2}\nabla\cdot(\rho_0\theta_0\mathbf{v}) + f_\pi, \tag{1.21}$$

where f_π consists of several small terms and can be neglected.

1.2.3 Anelastic and Boussinesq approximations

For cloud-scale motions, the mass conservation equation (1.10) can be approximated to a high degree of accuracy as

$$\nabla\cdot(\rho_o\mathbf{v}) = 0 \tag{1.22}$$

and the horizontal pressure gradient acceleration as

$$-\frac{1}{\rho_0}\nabla_H p^{'} \approx -\nabla_H \frac{p^{'}}{\rho_0}. \tag{1.23}$$

These expressions retain the variation of (basic state) density with height.

When the air motions are limited to a shallow layer, the density is approximately constant. Then ρ_0 can be replaced by a constant value in (1.22) and (1.23). The former becomes

$$\nabla \cdot \mathbf{v} = 0. \tag{1.24}$$

1.2.4 Pressure and temperature fluctuations

Equation (1.14) relates the fluctuations of density, pressure, and temperature:

$$\frac{\rho^{'}}{\rho_0} \approx \frac{p^{'}}{p_0} - \frac{T^{'}}{T_0}. \tag{1.25}$$

We now compare the magnitudes of the two terms on the right of (1.25). To do this, consider the magnitude of the horizontal pressure gradient that can be maintained within a "bubble" of gas characterized by pressure and temperature fluctuations $p^{'}$ and $T^{'}$. Using the momentum equation (1.13) in the x direction,

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = -\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial x},$$

we suppose that all terms on the left have similar magnitudes. If the bubble has a velocity scale U, then the order of magnitude of the pressure gradient acceleration is

$$-\frac{1}{\rho_0}\frac{\partial p^{'}}{\partial x} \sim U\frac{\partial U}{\partial x}.$$

Because ρ_0 depends only on z, we may write this as

$$-\frac{\partial}{\partial x}\left(\frac{p^{'}}{\rho_0}\right) \sim \frac{1}{2}\frac{\partial}{\partial x}U^2.$$

After integrating this, we can relate $p^{'}$ to U:

$$\frac{p^{'}}{\rho_0} \sim U^2.$$

Using the equation of state for dry air (1.2) applied to the hydrostatic base state, $p_0 = \rho_0 R_d T_0$, we find that

$$\frac{p^{'}}{p_0} \sim \frac{U^2}{R_d T_0} = \frac{c_p}{c_v}\frac{U^2}{c^2}, \tag{1.26}$$

where c is the speed of sound given by (1.16). Using (1.26) in (1.25), we see that the contribution of pressure fluctuations to density fluctuations, and therefore to buoyancy, may be neglected if

$$\frac{U^2}{c^2} \ll \frac{T^{'}}{T_0}.$$