A Simple Turbulence Closure Model

Atmospheric Sciences 6150

1 Cartesian Tensor Notation

Reynolds decomposition of velocity:
\[ \mathbf{V} = \mathbf{V} + \mathbf{v} \Rightarrow \mathbf{V} = U_i + u_i \]

Mean velocity:
\[ \overline{\mathbf{V}} = U_i + V_j + W_k = (U, V, W) \Rightarrow U_i = (U_1, U_2, U_3) \]

Turbulent velocity:
\[ \mathbf{v} = u_i + v_j + w_k = (u, v, w) \Rightarrow u_i = (u_1, u_2, u_3) \]

Gradient operator:
\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \Rightarrow \frac{\partial}{\partial x_k} = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \]

Advection operator:
\[ \nabla \cdot \nabla = U \frac{\partial}{\partial x} + V \frac{\partial}{\partial y} + W \frac{\partial}{\partial z} \Rightarrow U_k \frac{\partial}{\partial x_k} = U_1 \frac{\partial}{\partial x_1} + U_2 \frac{\partial}{\partial x_2} + U_3 \frac{\partial}{\partial x_3} \]

The covariance matrix is a tensor of rank 2:
\[
\begin{pmatrix}
 uu & uv & uw \\
 vv & vv & vw \\
 wu & wv & ww
\end{pmatrix} \Rightarrow u_i u_j = 
\begin{pmatrix}
 u_1 u_1 & u_1 u_2 & u_1 u_3 \\
 u_2 u_1 & u_2 u_2 & u_2 u_3 \\
 u_3 u_1 & u_3 u_2 & u_3 u_3
\end{pmatrix}
\]

Turbulent kinetic energy, \( e = q^2/2 \), and summation over repeated indices:
\[ q^2 = uu + vv + ww \Rightarrow q^2 = u_i u_i = u_1^2 + u_2^2 + u_3^2 \]
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November 15, 2007

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A tensor of rank 3:

\[
\begin{pmatrix}
uuu & uvu & uuw \\
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\end{pmatrix}, \begin{pmatrix}
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vuv & vvv & vvw \\
vwu & vwv & vww
\end{pmatrix}, \begin{pmatrix}
uuu & wvu & uwv \\
vuv & vvv & vww \\
wuv & wvw & www
\end{pmatrix}
\]

\[
\Rightarrow u_iu_ju_k = (u_1u_ju_k, u_2u_ju_k, u_3u_ju_k)
\]

\[
= \begin{pmatrix}
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u_2u_1u_1 & u_2u_1u_2 & u_2u_1u_3 \\
u_3u_1u_1 & u_3u_1u_2 & u_3u_1u_3
\end{pmatrix}, \begin{pmatrix}
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\end{pmatrix}
\]

Kronecker delta:

\[
\delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j
\end{cases}
\]

First moments of velocity (3 unique):

\[
U_i
\]

Second moments of velocity (9, 6 are unique):

\[
u_iu_j
\]

Third moments of velocity (27, ? are unique):

\[
u_iu_ju_k
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2 The Closure Problem

The momentum equation for a homogeneous incompressible fluid at high Re is

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla \left( \frac{p}{\rho_0} \right) + \nu \nabla^2 \mathbf{V}.
\] (1)

Decompose variables into means and deviations:

\[
\mathbf{V} = U_i + u_i
\]

\[
p = P + p
\]

Substitute into (1) and average:

\[
\frac{\partial U_j}{\partial t} + U_k \frac{\partial U_j}{\partial x_k} = -\frac{\partial}{\partial x_k} \left( \frac{p}{\rho_0} \right) - \frac{\partial u_k u_j}{\partial x_k} + \nu \nabla^2 U_j.
\] (2)

The additional term is due to momentum transport by the turbulent velocity fluctuations.

One way to close the equations is to assume that

\[
\bar{u_k u_j} = -K_m \left( \frac{\partial U_k}{\partial x_j} + \frac{\partial U_j}{\partial x_k} \right).
\]

This is the eddy viscosity model. However, \(K_m\) is a property of the flow, not of the fluid (as viscosity is), and is not necessarily a constant (as viscosity is). This type of model is generally poor in the atmosphere.

Another way to close the equations is to derive equations for the Reynolds stresses, \(u_k u_j\), and to make assumptions for the unknown terms in these equations in order to close the set of equations. This is a second-moment (or second-order) closure model.

3 Closure Models

4 A Simple Turbulence Closure Model

5 Richardson Number Dependence
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The momentum equation for a homogeneous incompressible fluid at high Re is

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3 Closure Models

First-order closure model:
\[
\begin{align*}
\{ u_i u_j \} & \Rightarrow \text{diagnostically related to} \left\{ U_i \right\} \\
\{ u_i \theta \} & \Rightarrow \text{prognostically determined}
\end{align*}
\]

Second-order closure model:
\[
\begin{align*}
\left\{ u_i u_j u_k \right\} & \Rightarrow \text{diagnostically related to} \left\{ \frac{u_i u_j}{\theta^2} \right\} \\
\left\{ p(u_i u_j + u_j u_i) \right\} & \Rightarrow \text{prognostically determined}
\end{align*}
\]

Third-order closure model:
\[
\begin{align*}
\left\{ u_i u_j u_k u_l \right\} & \Rightarrow \text{diagnostically related to} \left\{ \frac{u_i u_j u_k}{\theta^2} \right\} \\
\left\{ p((u_i u_j)_k + \ldots) \right\} & \Rightarrow \text{prognostically determined}
\end{align*}
\]
4 A Simple Turbulence Closure Model

We use the eddy viscosity model for the turbulent fluxes. We set the eddy viscosity $K_m$ to be proportional to the turbulence velocity scale $q$ times a turbulence length scale $l$. This allows $K_m$ to depend on the turbulence properties, which is a more realistic than using a constant $K_m$.

### Unknown turbulent fluxes

- **$\overline{u_i u_j}$ (momentum)**: $q^2 \delta_{ij} - ql_1 \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$
- **$\overline{u_i \theta}$ (any scalar)**: $-ql_2 \frac{\partial \Theta}{\partial x_i}$

To close these models for the turbulent fluxes, we require an equation for $q^2 \equiv \overline{u_i u_i}$. To do this, we start with the full equation for $q^2$:

$$\frac{dq^2}{dt} = -\frac{\partial \overline{u_i u_i}}{\partial x_j} - \frac{2 \rho \overline{u_i u_j}}{\partial x_j} - 2\overline{u_i u_j} \frac{\partial U_i}{\partial x_j} + 2g_i \frac{\Theta}{\overline{u_i u_i}} - 2\epsilon.$$  

The terms on the r.h.s. of this equation represent turbulent transport, pressure transport, shear (mechanical) production, buoyancy production (or loss), and dissipation, respectively. To close this equation we

1. Assume that production and dissipation balance:

$$0 = SP + BP - D.$$  

2. Use the models above for the fluxes.

3. Model dissipation using

$$\epsilon = \frac{q^3}{\Lambda_1}.$$  

The result is

$$q^2 = \Lambda_1 l_1 \left[ S_{ij} \frac{\partial U_i}{\partial x_j} - \frac{l_2 g_i}{\Theta} \frac{\partial \Theta}{\partial x_i} \right],$$  

where

$$S_{ij} \equiv \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}.$$
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<table>
<thead>
<tr>
<th>Unknown turbulent fluxes</th>
<th>Modeling assumption</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_i u_j$ (momentum)</td>
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To close these models for the turbulent fluxes, we require an equation for $q^2 \equiv \bar{u}_i \bar{u}_i$. To do this, we start with the full equation for $q^2$:

$$\frac{dq^2}{dt} = -\frac{\partial \bar{u}_i \bar{u}_j \bar{u}_j}{\partial x_j} - 2 \frac{\partial \bar{p} \bar{u}_j}{\partial x_j} - 2 \bar{u}_i \bar{u}_j \frac{\partial \bar{U}_i}{\partial x_j} + 2 \bar{g}_i \frac{\bar{u}_i \bar{\theta} v}{\bar{\Theta} \bar{\theta} v} - 2\epsilon.$$

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q^2 = \Lambda_1 l_1 \left[ S_{ij} \frac{\partial U_i}{\partial x_j} - \frac{l_2 g_i}{l_1 \Theta} \frac{\partial \Theta v}{\partial x_i} \right],
\]

where

\[
S_{ij} \equiv \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}
\]
Let $l_1 = A_1 l$, $l_2 = A_2 l$, and $\Lambda_1 = B_1 l$, where $l$ is the turbulence length scale. $A_1 = 0.92$, $A_2 = 0.74$, and $B_1 = 16.6$ are constants determined from experiments.

Many prescriptions for the turbulent length scale $l$ exist. The only definite constraint is that $l \to k z$ near the surface so that $K_m = k u_* z$ under neutral conditions. One commonly used form is

$$l = \frac{l_\infty}{1 + l_\infty/kz},$$

where the asymptotic length scale $l_\infty$ is specified to be about 10 percent of the boundary layer depth. The specification of $l$ is usually not very critical.

5 Richardson Number Dependence

This model includes the effects of stratification, so it should exhibit a dependence on Richardson number.

6 Performance

This simple closure model works best when its assumption of a local balance between production and dissipation is most nearly met. This condition is most likely to be valid when and where shear production dominates buoyancy production.
and

\[ g_i = (0, 0, g). \]

Let \( l_1 = A_1l, \ l_2 = A_2l, \) and \( \Lambda_1 = B_1l, \) where \( l \) is the turbulence length scale. \( A_1 = 0.92, \ A_2 = 0.74, \) and \( B_1 = 16.6 \) are constants determined from experiments. Many prescriptions for the turbulent length scale \( l \) exist. The only definite constraint is that \( l \to kz \) near the surface so that \( K_m = ku_z \) under neutral conditions. One commonly used form is

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5 Richardson Number Dependence

Equation (3) includes the effects of stratification, so it should exhibit a dependence on Richardson number. To show this, we will first simplify (3) by making the boundary layer approximation:

\[ U_3 = 0, \ \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2} = 0. \]

Then only

\[ S_{13} = S_{31} = \frac{\partial U_1}{\partial x_3} \]

and

\[ S_{23} = S_{32} = \frac{\partial U_2}{\partial x_3} \]

are nonzero, so

\[ S_{ij} \frac{\partial U_i}{\partial x_j} = \left( \frac{\partial U_1}{\partial x_3} \right)^2 + \left( \frac{\partial U_2}{\partial x_3} \right)^2. \]

Also,

\[ g_i \frac{\partial \Theta_v}{\Theta \partial x_i} = g \frac{\partial \Theta_v}{\Theta \partial x_3}. \]

Using these simplifications in (3), we obtain

\[ q^2 = \Lambda_1 l_1 \left[ \left( \frac{\partial U_1}{\partial x_3} \right)^2 + \left( \frac{\partial U_2}{\partial x_3} \right)^2 - \frac{l_2}{l_1} \frac{g \partial \Theta_v}{\Theta \partial x_3} \right]. \]
The condition for $q^2 > 0$ is therefore

$$\left(\frac{\partial U_1}{\partial x_3}\right)^2 + \left(\frac{\partial U_2}{\partial x_3}\right)^2 - \frac{l_2}{l_1} g \frac{\partial \Theta}{\partial x_3} > 0.$$ 

Write this in terms of a gradient Richardson number $R_i$:

$$\frac{l_1}{l_2} > \frac{g \frac{\partial \Theta}{\partial x_3}}{\left(\frac{\partial U_1}{\partial x_3}\right)^2 + \left(\frac{\partial U_2}{\partial x_3}\right)^2} \equiv R_i$$

or

$$R_i < \frac{l_1}{l_2} = \frac{A_1 l}{A_2 l} = \frac{A_1}{A_2} = \frac{0.92}{0.74} = 1.24.$$ 

Theoretical and laboratory results suggest that laminar flow becomes turbulent when

$$R_i < 0.25,$$

and that turbulent flow becomes laminar when

$$R_i > 1.$$ 

These are local criteria. Even if $R_i > 1$ when estimated using resolved variables, it may be $< 1$ locally within a grid volume.

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