Spectral Analysis

The goal of *spectral estimation* is to describe the distribution (over frequency) of the power contained in a signal, based on a finite set of data. Estimation of power spectra is useful in a variety of applications, including the detection of signals buried in wide-band noise.

The *power spectrum* of a stationary random process x_n is mathematically related to the correlation sequence by the discrete-time Fourier transform. In terms of normalized frequency, this is given by

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} R_{xx}(m)e^{-j\omega m}$$

This can be written as a function of physical frequency f (e.g., in hertz) by using the relation $\omega = 2\pi f/f_s$, where f_s is the sampling frequency.

$$S_{xx}(f) = \sum_{m=-\infty}^{\infty} R_{xx}(m)e^{-2\pi jfm/f_s}$$

The correlation sequence can be derived from the power spectrum by use of the inverse discrete-time Fourier transform:

$$R_{xx}(m) = \int_{-\pi}^{\pi} \frac{S_{xx}(\omega)e^{j\omega m}}{2\pi}d\omega = \int_{-f_s/2}^{f_s/2} \frac{S_{xx}(f)e^{2\pi jfm/f_s}}{f_s}df$$

The average power of the sequence x_n over the entire Nyquist interval is represented by

$$R_{xx}(0) = \int_{-\pi}^{\pi} \frac{S_{xx}(\omega)}{2\pi}d\omega = \int_{-f_s/2}^{f_s/2} \frac{S_{xx}(f)}{f_s}df$$

The quantities

$$P_{xx}(\omega) = \frac{S_{xx}(\omega)}{2\pi} \qquad \text{and} \qquad P_{xx}(f) = \frac{S_{xx}(f)}{f_s}$$

from the above expression are defined as the *power spectral density* (PSD) of the stationary random signal x_n.

The average power of a signal over a particular frequency band $[\omega_1, \omega_2]$, $0 \le \omega_1 < \omega_2 \le \pi$, can be found by integrating the PSD over that band:

$$\bar{P}_{[\omega_1, \omega_2]} = \int_{\omega_1}^{\omega_2} P_{xx}(\omega)d\omega + \int_{-\omega_2}^{-\omega_1} P_{xx}(\omega)d\omega$$

You can see from the above expression that $P_{xx}(\omega)$ represents the power content of a signal in an *infinitesimal* frequency band, which is why we call it the power spectral *density*.

The units of the PSD are power (e.g., watts) per unit of frequency. In the case of $P_{xx}(\omega)$, this is watts/rad/sample or simply watts/rad. In the case of $P_{xx}(f)$, the units are watts/hertz. Integration of the PSD with respect to frequency yields units of watts, as expected for the average power $\bar{P}_{[\omega_1,\omega_2]}$.

For real signals, the PSD is symmetric about DC, and thus $P_{xx}(\omega)$ for $0 \le \omega < \pi$ is sufficient to completely characterize the PSD. However, in order to obtain the average power over the entire Nyquist interval it is necessary to introduce the concept of the *one-sided* PSD.

The one-sided PSD is given by

$$P_{onesided}(\omega) = \begin{cases} 0, & -\pi \le \omega < 0 \\ 2P_{xx}(\omega), & 0 \le \omega < \pi \end{cases}$$

The average power of a signal over the frequency band $[\omega_1,\omega_2]$, $0 \le \omega_1 < \omega_2 \le \pi$, can be computed using the one-sided PSD as

$$\bar{P}_{[\omega_1,\omega_2]} = \int_{\omega_1}^{\omega_2} P_{onesided}(\omega)d\omega$$