Fundamental Math Concepts

Atmos 5110 Synoptic–Dynamic Meteorology I

Instructor: Jim Steenburgh

jim.steenburgh@utah.edu

801-581-8727

Suite 480/Office 488 INSCC

Suggested reading: Lackmann (2011), section 1.2

These are essential math concepts for success in this and other atmospheric sciences classes. To do well in this class, you must understand these concepts mathematically and physically.

Cartesian Coordinates

A coordinate system based on the orthogonal (i.e., perpendicular) unit vectors $\hat{\imath}$ (x, postive eastward), $\hat{\jmath}$ (positive northward), and $\hat{k}$ (positive upward). Pressure, p, and potential temperature, θ, are sometimes used instead of height.

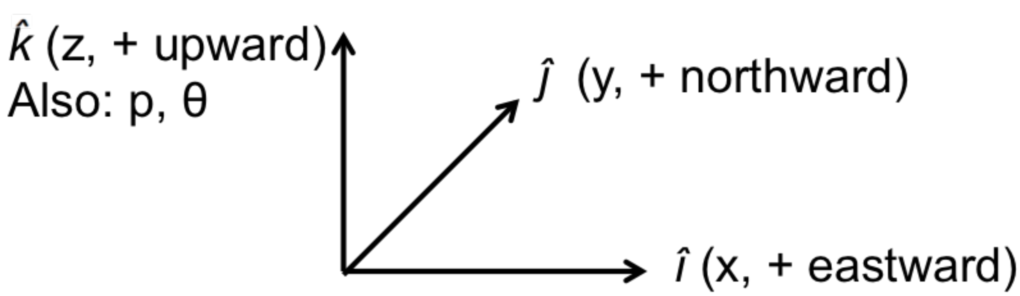


Figure adapted from Lackmann (2011)

Gradient (a.k.a. "del") Operator

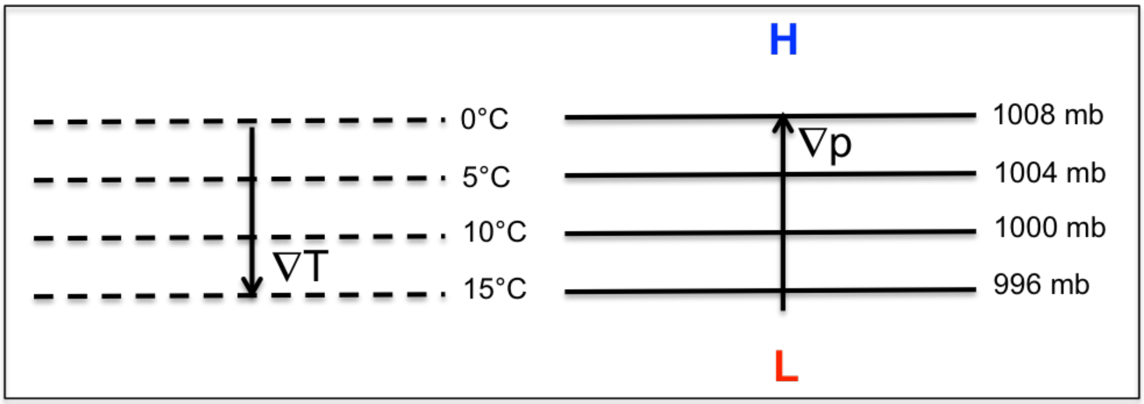
The three dimensional rate of change of a quantity with distance

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

Example (temperature)

$$\nabla \mathrm{T} = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath} + \frac{\partial T}{\partial z}\hat{k}$$

The ∇ is a vector that always points toward *higher* values.



Often in synoptic meteorology we consider only the two-dimensional horizontal gradient. For example,

$$\nabla_z \mathrm{T} = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath}$$

or

$$\nabla_p \mathrm{T} = \frac{\partial T}{\partial x}\hat{\imath} + \frac{\partial T}{\partial y}\hat{\jmath}$$

where the *z* and *p* subscripts indicate the gradient on a height or pressure surface, respectively.

<u>Laplacian Operator</u>

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Example

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Often in synoptic meteorology we consider only the two-dimensional Laplacian. For example

$$\nabla_z^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

or

$$\nabla_p^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

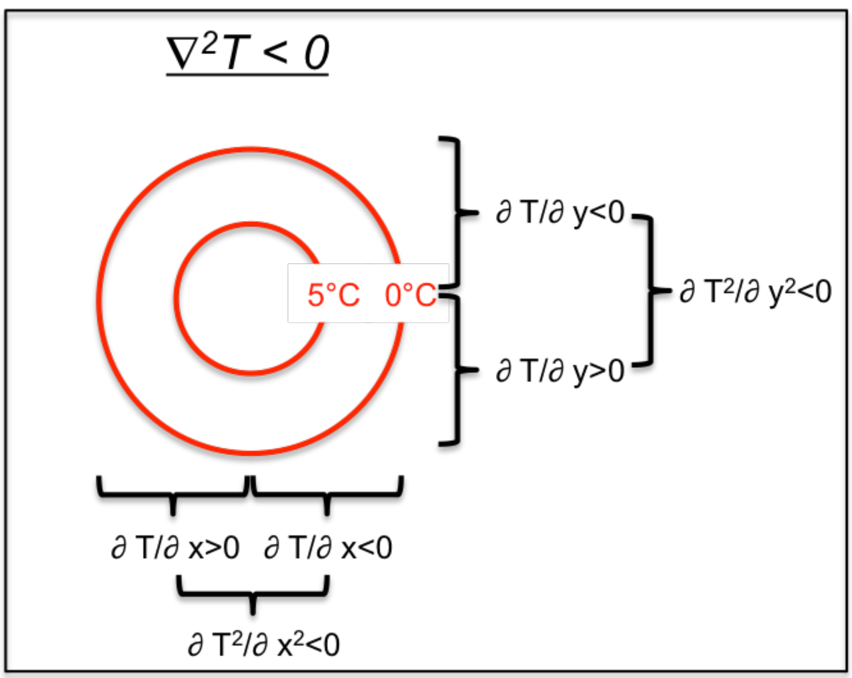
Synoptic interpretation

A <u>minimum</u> in T $\Rightarrow \nabla^2 T > 0$

A <u>maximum</u> in T $\Rightarrow \nabla^2 T < 0$

Therefore we sometimes approximate ∇^2 as a minus sign (i.e., $\nabla^2 \sim -$)

Synoptic example (2D Laplacian)



<u>Dot Product</u>

Given two vectors

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \qquad and \qquad \vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$$

then

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Example

$$\nabla\cdot\vec{U} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(u\hat{\imath} + v\hat{\jmath} + w\hat{k}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \rightarrow Divergence!$$

The dot product is a *scalar.*

"$\nabla\cdot$" is known mathematically as the divergence.

Note: Following Lackmann (2011), we will use $\vec{U}$ for the 3-D velocity vector and $\vec{V}$ for the 2-D velocity vector. Holton and Hakim (2013) use $\boldsymbol{U}$ an $\boldsymbol{V}$.

<u>Cross Product</u>

Given two vectors

$$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k} \qquad and \qquad \vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$$

then

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \left(a_y b_z - b_y a_z\right)\hat{\imath} - (a_x b_z - b_x a_z)\hat{\jmath} + (a_x b_y - b_x a_y)\hat{k}$$

Example

$$\nabla\times\vec{U} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \rightarrow Vorticity!$$

The cross product is a *vector* with a magnitude and a direction.

"$\nabla\times$" is known mathematically as the curl.

Expansion of a Total Derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{U} \cdot \nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

Total Derivative
Lagrangian rate of change
Derivative following a parcel
Variable is conserved if D/Dt=0

Partial Derivative
Eulerian rate of change
Derivative at a fixed location
Steady state conditions if ∂/∂ t=0

"-" Advection
Transport by the flow
(Advection is $-U\cdot\nabla$)

Example

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{U} \cdot \nabla \mathrm{T} \; = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Or in pressure coordinates

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{U} \cdot \nabla \mathrm{T} \; = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}$$

where $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}$ is evaluated on a pressure surface and is ω is the vertical velocity in pressure coordinates, Dp/Dt.

Alternatively, we can rearrange to show the rate of change at a given location

$$\frac{\partial}{\partial t} = \frac{D}{Dt} - \vec{U} \cdot \nabla$$

Local Change = Change following the flow + Advection

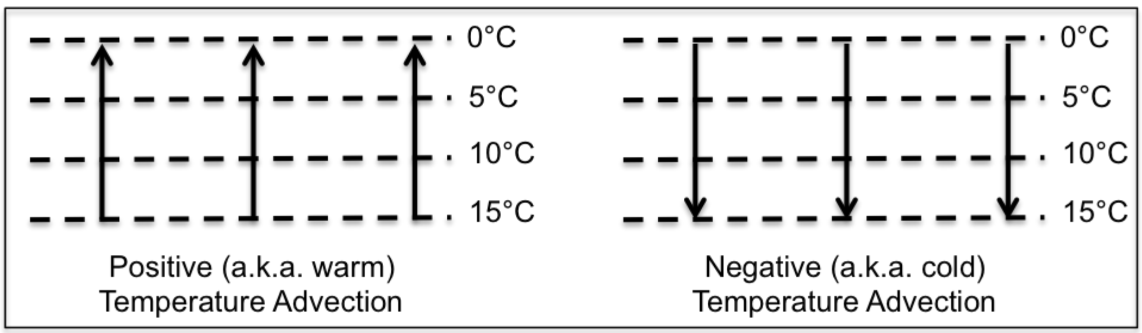
Example

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \vec{U} \cdot \nabla T$$

Diagnosing Advection

Advection is the transport of a quantity by the flow, such as the transport of air pollution, temperature, or vorticity. Although three dimensional, often in synoptic meteorology, we consider only the horizontal advection (i.e., $-\vec{V} \cdot \nabla T$).

- Positive advection (i.e., $-\vec{V} \cdot \nabla T > 0$) indicates that larger concentrations or values are being transported into a region of lower values.
- Negative advection (i.e., $-\vec{V} \cdot \nabla T < 0$) indicates that lower concentrations or values are being transported into a region of higher values.



The strength of the advection depends on:

1. The strength of the wind
2. The strength of the gradient in the quantity of interest
3. The angle of the wind relative to the isolines of the quantity of interest.

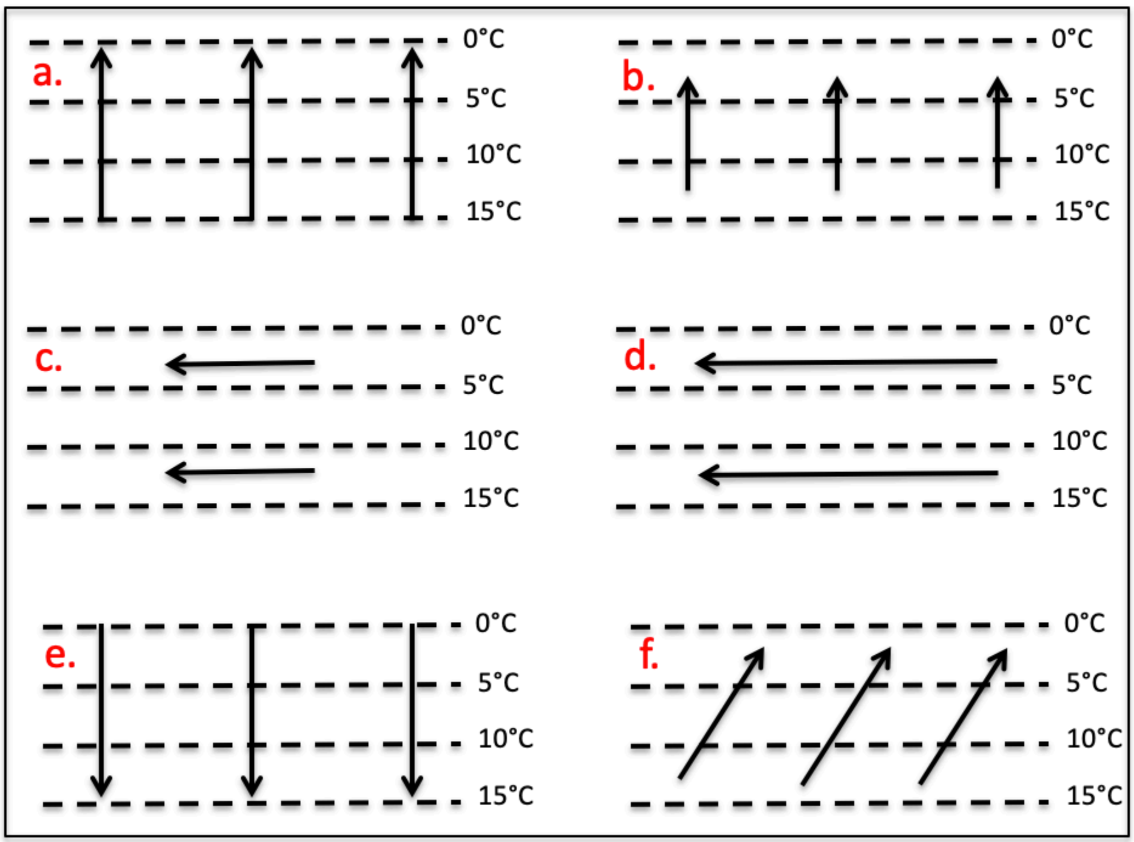
Review (using Classquestion)

Q1: See classquestion.com

Q2: Which of the figures below features the strongest warm-air advection?

Q3: Which of the figures below features cold-air advection?

Q4: Which of the figures below features no temperature advection and the strongest flow?



Q5: Which of the following is the mathematical expression for advection?

a. $\vec{U} \cdot \nabla$

b. $-\vec{U} \cdot \nabla$

c. $\vec{U}\nabla$

d. $\vec{U} \times \nabla$

Q6–Q9: See Classquestion

<u>Class activity</u>

Using the IDV Supernational bundle, diagnose the sign and relative strength of the 925-mb temperature advection and the 500-mb geostrophic vorticity advection over (a) Seattle, WA, (b) Salt Lake City, UT, and (c) Albany, NY.