Scale Analysis of the Vorticity Equation

Atmos 5110 Synoptic–Dynamic Meteorology I

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Suggested reading: Lackmann (2011), section 1.3

Holton and Hakim (2013), section 4.3.3

Scale analysis applied to the vorticity equation helps us to isolate the key dynamical processes and the influence of planetary vorticity on the evolution of large-scale weather systems.

Scale analysis of the vortcity equation

Reading: See Lackman (2011) section 1.3

Motivation: Scale analysis allows us to evaluate what terms are important for the evolution of large scale weather systems

Typical magnitudes for synoptic systems

Description	**Symbol**	**Magnitude**
Horizontal velocity scale	U	10 m s^{-1}
Vertical velocity scale	W	10-2 m s^{-1}
Length scale	L	10^6 m (1000 km)
Depth scale	H	10^4 m (10 km)
Horizontal pressure change	Δp_h	10^3 Pa (10 mb)
Density	ρ	1 kg m^{-3}
Time	T=L/U	10^5 s (27.8 h)
Coriolis	f	10^{-4} s^{-1}
Beta effect	$\beta=\partial f/dy$	10^{-11} m^{-1} s^{-1}

Normalized horizontal density change	Δ ρ/ ρ	10^{-2}

Recall the vorticity equation

$$\frac{D(\zeta+f)}{Dt} = -(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

1. For the synoptic scale, how large is the relative vorticity compared to the planetary vorticity?

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \sim \frac{U}{L} \sim \frac{10\ m\ s^{-1}}{10^6\ m} \sim 10^{-5}\ s^{-1}$$

$$\frac{|\zeta|}{|f|} \sim \frac{U/L}{f} \sim \frac{U}{fL} = Rossby\ Number\ (R_0)$$

$$\frac{|\zeta|}{|f|} \sim \frac{10^{-5}\ s^{-1}}{10^{-4}\ s^{-1}} \sim 0.1 \ll 1$$

Conclusion: For synoptic-scale systems, the relative vorticity is an order of magnitude smaller than the planetary vorticity $\Longrightarrow \zeta$ can be neglected in the divergence term.

Note: Flows with $R_0 \ll 1$ are called low-Rossby number flows. Low Rossby number flows are near geostrophic and, because $f >> \zeta$, convergence increases absolute vorticity and divergence decreases absolute vorticity (in NH).

2. What is the magnitude of the Divergence Term?

$$\left|-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right| \sim \left|f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right| \sim f\frac{U}{L} \sim 10^{-4} s^{-1} \cdot \frac{10\ m\ s^{-1}}{10^6\ m} = 10^{-9} s^{-2}$$

However, $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are usually of opposite signs but nearly equal magnitude, therefore

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \neq \frac{U}{L}\ (10^{-5} s^{-1}), but\ is\ usually\ closer\ to\ 10^{-6} s^{-1}$$

$$\left|-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \sim 10^{-10} s^{-2}$$

2. What is the magnitude of the Tilting Term?

$$\left|-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)\right| \sim \frac{W}{L}\cdot\frac{U}{H} \sim \frac{10^{-2}\ ms^{-1}}{10^{6}\ m}\cdot\frac{10\ ms^{-1}}{10^{4}\ m} \sim 10^{-11} s^{-2}$$

⇒ On the synoptic scale, the tilting term is an order of magnitude smaller than the divergence term and can be neglected.

3. What is the magnitude of the Solenoidal term?

$$\left|\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)\right| \sim \frac{1}{\rho}\cdot\frac{\Delta\rho}{\rho L}\cdot\frac{\Delta p}{L} \sim \frac{1}{1kgm^{-3}}\cdot\frac{10^{-2}}{10^{6}m}\cdot\frac{10^{3}Pa}{10^{6}m} \sim 10^{-11} s^{-2}$$

⇒ On the synoptic scale, the solenoidal term is also an order of magnitude smaller than the divergence term and can be neglected.

4. What is the magnitude of the terms in $\frac{D(\zeta+f)}{Dt}$?

$$\frac{D(\zeta + f)}{Dt} = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial t}, u\frac{\partial f}{\partial x}, w\frac{\partial f}{\partial z} = 0$$

$$\frac{D(\zeta + f)}{Dt} = \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + v\frac{\partial f}{\partial y}$$

$$\left|\frac{\partial \zeta}{\partial t}\right|, \left|u\frac{\partial \zeta}{\partial x}\right|, \left|v\frac{\partial \zeta}{\partial y}\right| \sim \frac{U^2}{L^2} \sim \frac{(10\ ms^{-1})^2}{(10^{6}m)^2} \sim 10^{-10} s^{-2} \Rightarrow Keep$$

$$v\frac{\partial f}{\partial y} \sim U\beta \sim (10\ ms^{-1})(10^{-11}m^{-1}s^{-1}) \sim 10^{-10} s^{-2} \Rightarrow Keep$$

$$w\frac{\partial \zeta}{\partial z} \sim \frac{WU}{LH} \sim \frac{(10^{-2}\ ms^{-1})(10\ ms^{-1})}{(10^{6}m)(10^{4}m)} \sim 10^{-11} s^{-2} \Rightarrow Neglect$$

⇒ On the synoptic scale, the total absolute vorticity advection can be approximated by the horizontal advection

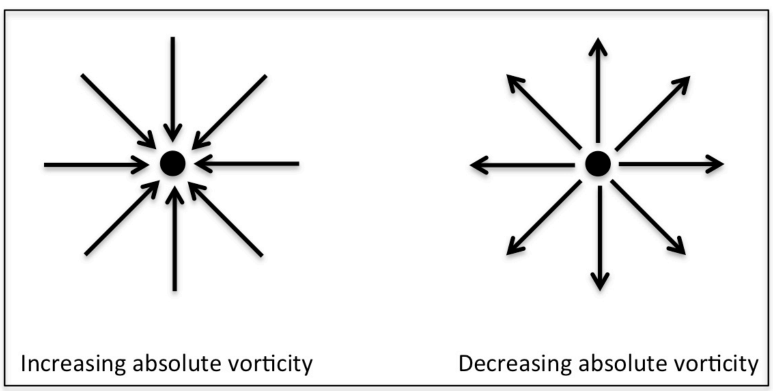
$$\frac{D(\zeta+f)}{Dt} \sim \frac{D_h(\zeta+f)}{Dt} = \frac{\partial(\zeta+f)}{\partial t} + u\frac{\partial(\zeta+f)}{\partial x} + v\frac{\partial(\zeta+f)}{\partial y}$$

and we can write an approximate form of the vorticity equation for synoptic-scale motion as:

$$\frac{D_h(\zeta+f)}{Dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Synoptic interpretation:

- For synoptic-scale flows, the tilting term, solenoidal term, and vertical advection of vorticity can be neglected
- f plays a dominant role in the divergence term (i.e., f > ζ), therefore changes in absolute vorticity following the flow are produced by horizontal convergence (increasing $\zeta + f$) and divergence (decreasing $\zeta + f$)



Note: Holton and Hakim (2013) argue that for intense cyclonic storms, the ζ should be retained in the divergence term

$$\frac{D_h(\zeta+f)}{Dt} = -(\zeta+f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$