Once droplets are activated, how do they grow? There are two primary components to this problem. Growth by vapor diffusion, and growth by collision-coalescence.

We showed that one of many legitimate forms for the non-equilibrium solution for flows from high to low potential is a current $\vec{j}$ that adds to the amount of mass $m$ in a lower potential surface at constant $T$ and $p$ through the equation

$$\vec{j} = \left( \frac{\partial m}{\partial t} \right)_{T,p} = -\frac{A}{RT} D \nabla p = -AD\nabla \rho$$

where, the sign convention here is for the direction to be from the high potential to the low potential, so that a negative density gradient in this direction corresponds with positive flow.

Well, of course this could apply to the diffusional growth of a droplet, where the droplet is a low potential surface, and its super-saturated environment is a higher potential surface. Evaluated at the droplet surface, where the droplet has radius $r$, the mass increase $m$ in condensed molecules, at the expense of vapor $m_v$, and to surface area $A = 4\pi r^2$ follows

$$\left( \frac{\partial m_v}{\partial t} \right)_{T,p} = \left( \frac{\partial m_d}{\partial t} \right)_{T,p} = 4\pi r^2 D \frac{\partial \rho}{\partial x} \bigg|_{x=r}$$

where, the diffusivity $D$ for the water vapor in air is about $1.5 \times 10^{-5}$ m$^2$ s$^{-1}$. Also, here we’ve switched the direction $x$ so that it points radially away from the droplet.

But, what is $\partial \rho/\partial x|_{x=r}$? If we assume the ambient vapor field is in steady-state over time scales relevant to instantaneous flows, which is extremely close to being true, then

$$d\rho/dt = \nabla^2 \rho = 0$$

If we assign $x$ as the radial dimension, in spherical coordinates, we get

$$\frac{\partial^2 \rho}{\partial x^2} + \frac{1}{2x} \frac{\partial \rho}{\partial x} = 0$$

which has the general solution

$$\rho(x) = C_1 - C_2/x$$

Applying the boundary conditions $x \rightarrow \infty, \rho \rightarrow \rho_\infty$; $x \rightarrow r, \rho \rightarrow \rho_r$, we find the vapor field can be described by

$$\rho(x) = \rho_\infty - \frac{r}{x} (\rho_\infty - \rho_r)$$

Thus, taking our flux solution,

$$\frac{dm}{dt} = 4\pi r^2 D \frac{\partial \rho}{\partial x} \bigg|_{x=r}$$

$$\frac{dm}{dt} = 4\pi r D (\rho_\infty - \rho_v)$$

So rate of mass accumulation of droplets is proportional to their size. Note that the above equation is for a single droplet only, and that we would need to multiply the above equation by the droplet number concentration to get total mass density for a cloud. What stops droplets from
growing indefinitely? Naturally, as droplets grow, they will deplete the available water from their surroundings, thereby reducing the vapor concentration gradient between the ambient air and the surface of the droplet.

How fast do droplets grow in size? Starting from \( m = \frac{4}{3} \pi \rho r^3 \)

\[
\frac{dm}{dt} = 4\pi \rho r^2 \frac{dr}{dt}
\]

\[
\frac{dr}{dt} = \frac{D}{r \rho} (\rho_{v\infty} - \rho_{vr})
\]

Using the ideal gas law for water vapor

\[
\frac{dr}{dt} = \frac{D}{r \rho_1 R_v T} (e_\infty - e_r)
\]

\[
\frac{dr}{dt} = \frac{D \rho_{v\infty}}{r \rho_1 e_\infty} (e_\infty - e_r)
\]

which, assuming \( e_\infty \approx e^{sat} (T) \), such that the perturbation is not large, we can show that

\[
\frac{e_\infty - e_r}{e_\infty} \approx \frac{e_\infty - e_s}{e_s} \equiv S - 1 = s
\]

where \( S \) is the saturation ratio and \( s \) is the supersaturation. So

\[
r \frac{dr}{dt} = G_t s
\]

where

\[
G_t = \frac{D \rho_{v\infty}}{\rho_1} = \frac{D e^{sat} (T)}{\rho_1 R_v T}
\]

We have made a number of assumptions in these derivations that we should be aware of.

1. All water vapor molecules that hit the droplet stick. The reality may be that only 3% do. This matters mostly during initial growth of the droplet.

2. The temperature at the surface of the droplet is the same as the ambient air. In reality it is warmer due to the latent heat associated with condensation. This retards growth.

3. The droplet is stationary with respect to its environment. A droplet that falls as it grows is “ventilated” which changes the distribution of water vapor around the droplet and carries away latent heat. Apparently, however, this effect is very small, and can usually be neglected.

If we integrate the growth rate equation we get

\[
r (t) = \sqrt{r_0^2 + 2G_t st}
\]

where \( r_0 = r^* \), the activation radius if we are starting from a haze particle. A table of the amount of time it takes for a particle to reach a given size is given below. Note that droplets grow very rapidly at first, and as the droplets grow their initial size matters less and less.
TABLE 7.2. Rate of Growth of Droplets by Condensation (initial radius 0.75 µm). (From Mason, 1971)

<table>
<thead>
<tr>
<th>Nuclear mass (g)</th>
<th>$10^{-14}$</th>
<th>$10^{-13}$</th>
<th>$10^{-12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius (µm)</td>
<td>Time (sec) to grow from initial radius 0.75 µm</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.4</td>
<td>0.15</td>
<td>0.013</td>
</tr>
<tr>
<td>2</td>
<td>130</td>
<td>7.0</td>
<td>0.61</td>
</tr>
<tr>
<td>4</td>
<td>1,000</td>
<td>320</td>
<td>62</td>
</tr>
<tr>
<td>10</td>
<td>2,700</td>
<td>1,800</td>
<td>870</td>
</tr>
<tr>
<td>20</td>
<td>8,500</td>
<td>7,400</td>
<td>5,900</td>
</tr>
<tr>
<td>50</td>
<td>17,500</td>
<td>16,000</td>
<td>14,500</td>
</tr>
<tr>
<td>50</td>
<td>44,500</td>
<td>43,500</td>
<td>41,500</td>
</tr>
</tbody>
</table>

Figure 1: The assumed supersaturation is 0.05%, $p = 900$ mb and $T = 273K$.

TABLE 7.3. Distance a Drop Falls before Evaporating, assuming Isothermal Atmosphere with $T = 280K$, $S = 0.8$

<table>
<thead>
<tr>
<th>Initial radius</th>
<th>Distance fallen</th>
</tr>
</thead>
<tbody>
<tr>
<td>1µm</td>
<td>2µm</td>
</tr>
<tr>
<td>3µm</td>
<td>0.17mm</td>
</tr>
<tr>
<td>10µm</td>
<td>2.1cm</td>
</tr>
<tr>
<td>30µm</td>
<td>1.69m</td>
</tr>
<tr>
<td>0.1mm</td>
<td>208 m</td>
</tr>
<tr>
<td>0.15 mm</td>
<td>1.05 km</td>
</tr>
</tbody>
</table>

Figure 2: Droplet evaporation
The same equations can be used to estimate the evaporation rate of droplets below cloud base. It is easy to see from this table why the edges of clouds are so sharp. Only falling drops larger than about 0.1 mm will give any blurring to the cloud, and droplets must be significantly bigger in order to have any chance of hitting the ground, particularly if cloud base is very high (as it often is here in Utah in Summer).

It is important to recognize that when a water drop evaporates, it does not completely disappear, but rather becomes a haze particle whose size is determined by the Kohler equation according to the mass of solute it contains, and the ambient relative humidity (aka saturation ratio).