Mixing and Turbulence

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This section introduces some elementary concepts associated with mixing and turbulence in the environment.

1 Conserved Variables

Studies of mixing of different airmasses often approach the problem by considering the conserved variables associated with each. Thus far we have discussed several of these

1. Conserved under unsaturated adiabatic processes

   • \( \theta, h_d, w \)

2. Conserved under saturated adiabatic processes

   • \( \theta_e, h_m, Q \)

Note that \( w_s \) and \( \chi \) are not conserved in general, and variables that are conserved under dry processes are not conserved under moist processes.

Now suppose that we have two airmasses that meet and mix, without interaction with the surrounding environment. Such mixing is a fundamental aspect of for example heat transport from the equator to the poles, or temperature and humidity gradients along fronts, or entrainment of dry air into clouds. How do we determine the state variables associated with the mixed parcels?

By way of illustration if we take mass fraction \( f \) from one air parcel \( A \) and mass fraction \( (1 - f) \) from a second parcel \( B \) and there is no saturation during the mixing process then the dry static energy and mixing ratio associated with the new parcel is

\[
h_d = fh_{dA} + (1 - f) h_{dB}
\]
\[ w = fw_A + (1 - f)w_B \]

Since \( h_d = c_pT + gz \), if one knows the height of the new parcel, one now knows the temperature and humidity of the new parcel.

## 2 Adiabatic Mixing without Condensation

We derive here the basic equations dealing with adiabatic mixing without condensation.

**Assumptions**

- The two parcels with masses \( m_1 \) and \( m_2 \) mix adiabatically
- The two parcels are at the same level \((d(gz) = 0)\) (e.g. mixing across a front).

Since the two parcel are at the same level we are mixing only enthalpy and water vapor. Therefore,

\[ m_1 h_1 + m_2 h_2 = (m_1 + m_2) h \]

The change in enthalpy is then

\[ \Delta H = 0 = m_1 (h - h_1) + m_2 (h - h_2) \]

\[ m_1 \Delta h_1 + m_2 \Delta h_2 = 0 \]

\[ m_1 c_p (T - T_1) + m_2 c_p (T - T_2) = 0 \]

so the new temperature is

\[ T = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2} \]

Similarly

\[ w = \frac{m_1 w_1 + m_2 w_2}{m_1 + m_2} \]

\[ e = \frac{m_1 e_1 + m_2 e_2}{m_1 + m_2} \]

\[ \theta = \frac{m_1 \theta_1 + m_2 \theta_2}{m_1 + m_2} \]
**Example**

Two parcels of air mix thoroughly across a cold front at 1000 mb. Parcel one has a temperature of 23.8°C and $w = 16 \text{ g/kg}$, and parcel two has a temperature of $T = 12.4\text{ C}$ and a mixing ratio of 5 g/kg.

1. If both parcels of air mix equally, what is the final temperature and mixing ratio of the combined air masses?

   Since the parcels mix equally, the new temperature and mixing ratio is simply the average temperature and mixing ratio, which are 18.1 and 10.5 g/kg.

2. What is the initial RH of each parcel and final RH?

   It works out that
   
   $$e_s(23.8^\circ) = 29.5 \text{ mb}$$
   $$e_s(12.4^\circ) = 14.4 \text{ mb}$$
   $$e_s(18.1^\circ) = 20.7 \text{ mb}$$

   and that the saturated mixing ratio is
   
   $$w_s = 0.622 \frac{e_s}{p}$$
   $$w_s(23.8^\circ) = 0.0183$$
   $$w_s(12.4^\circ) = 0.009$$
   $$w_s(18.1^\circ) = 0.0129$$

   So the relative humidities are
   
   $$RH(23.8^\circ) = 87\%$$
   $$RH(12.4^\circ) = 56\%$$
   $$RH(18.1^\circ) = 82\%$$

   So unlike $T$ and $w$, $RH$ does not mix linearly! This is an important consideration since it leads to the observation that two subsaturated parcels of air can combine to produce a cloud.
Figure 1: Two subsaturated parcels combining to produce a saturated parcel. Note that both $e$ and $T$ mix linearly, but that clearly $RH$ does not.

A most striking example of this sort of this is when a cold dry air mass moves over a warm moist air or surface to produce sea-smoke.
Another example is when warm moist exhaust from an airplane mixes with cold dry air to produce a contrail.

3 Turbulence

3.1 Characteristics

Examples of Turbulence

- Cumulus clouds
- Jet streams
- Dust storms

Characteristics

- irregularity
- diffusivity - no spreading no turbulence
- vorticity
• dissipative due to viscous losses (waves are non-dissipative although they can be dispersive)

• characteristic of fluid flows

• Associated with large values of the Reynolds number $Re = \frac{uL}{\nu}$, where $L$ is the characteristic length scale of an object moving with speed $u$ through a viscous medium with kinematic viscosity $\nu$.

Sources

• shear in a mean flow

• buoyancy

The transition from laminar to turbulent flow is one of the more poorly understood aspects of physics. On his death bed, the famous fluid dynamicist Horace Lamb said, "I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic"

3.2 Length Scales in Turbulent Flows

Turbulent flows have a wide range of length scales covering many orders of magnitude in size ranging from the dimensions of the flow itself at the high end, to the length scale associated with the diffusive action of molecular viscosity at the low end.

3.2.1 Laminar Boundary Layers

What are the relevant scales for laminar, low-shear flow? The Navier-Stokes equations for steady flow are

$$u \frac{du}{dx} = -\frac{1}{\rho} \frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial x^2}$$

The first term is inertia, the second the pressure gradient force, and there is a non-linear third term, the viscous term with $\nu = 0.15 \text{ cm}^2 \text{ s}^{-1}$, which is needed to create turbulence. For viscosity to be sufficiently large to damp turbulence the viscous and inertia terms have to be of the same magnitude. Doing scale analysis we would require

$$\frac{U^2}{L} \sim \frac{\nu U}{L^2}$$
Dividing the first term by the second, we obtain the Reynolds number

\[ \frac{UL}{\nu} \sim \text{Re} \]

If the Reynolds number is of order unity than turbulence is damped and flow is laminar. But Reynolds numbers typical of the atmosphere are typically on the order of a few thousand, so the atmosphere tends to be turbulent.

In clouds the source of energy is mostly latent heat release, and this creates turbulent motions while sending air upwards. With respect to the turbulence, it is common to talk about an energy dissipation rate \( \epsilon \) with units of energy per mass per second \( (\text{m}^2/\text{s}^3) \). It represents the rate at which energy is transferred from large-scale motion to small scales. Eddies start big and then become progressively smaller.

But how small before the flows become laminar because viscosity takes over? Combining the rate at which the smallest scales receive energy with the viscosity, using dimensional relationships we can derive length, time, and velocity scales for the smallest scales as follows

\[ \eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/4} \]

\[ \tau = (\nu/\epsilon)^{1/2} \]

\[ v = (\nu\epsilon)^{1/4} \]

These are called the Komolgorov microscales of length, time, and velocity. The corresponding Reynolds number formed with these numbers is equal to one

\[ \eta v/\nu = 1 \]

Small-scale motion is highly viscous, and that the viscous dissipation of turbulence as heat adjusts itself to the energy supply \( \epsilon \) by adjusting length scales \( \eta \).

Can we estimate what the supply rate of energy \( \epsilon \) is to small scales? Well the energy comes from large scale turbulence. This energy is proportional to \( u^2 \). It seems appropriate to assume that the time scale \( T \) associated with dissipation of the energy is then \( u/l \) which is roughly one overturning of the eddy. Therefore the rate of energy dissipation is of order \( u^2/T = u^2 \cdot u/l \).

\[ \epsilon \sim u^3/l \]

Let’s say you measure a time series of wind speed. By doing a fourier transform on this time series you can obtain a power spectrum of the data. It works out that for there to be a cascade
of energy from large scales to small scales requires that the slope of this energy spectrum be $-5/3$. At one end of the spectrum are the eddies of speed $u$ with length scale $l$. At the other end is the Kolmogorov microscale with speed and length scales $v$ and $\eta$. The rate at which energy is transferred between the two is $\epsilon$, and this is also the rate at which large scale turbulence gets converted to heat. The power spectrum continues leftward to larger length scales, but these tends to be associated with gravity waves first, and synoptic scale waves second.

**Example**

Estimate the energy dissipation rate in a cumulus cloud, both per unit mass and for the entire cloud. Base your estimates on velocity and length scales typical of cumulus clouds. Compute the total dissipation rate in kilowatts. Also estimate the Kolmogorov microscale $\eta$. Use $\rho = 1.25 \text{ kg/m}^3$ and $\nu = 15 \times 10^{-6} \text{ m}^2/\text{s}$.

![A cumulus cloud](image)

We can estimate the dissipation rate using

$$\epsilon \sim u^3/l$$

In a cumulus cloud a typical turret (arguably the length scale of the largest eddies) is about 250 m. The updraft velocity is about 1 m/s. This gives values of $\epsilon \sim 4 \times 10^{-3} \text{ m}^2/\text{s}^3$ for the energy dissipation rate per unit mass. The energy dissipation rate of the entire cloud, assuming fairly
Figure 3: Analysis of turbulence data
suitable dimensions of 1 km$^3$ and $\rho = 1.25$ kg/m$^3$ is

\[ m \times \epsilon = 4 \times 10^{-3} \times 1.25 \times 1 \times 10^9 = 5 \times 10^6 \text{J s}^{-1} = 5000 \text{KW} \]

(How does this compare to energy production associated with latent heat release during ascent?). The Kolmogorov microscale is

\[ \eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/4} = \left( \frac{(15 \times 10^{-6})^3}{4 \times 10^{-3}} \right)^{1/4} = 1 \times 10^{-3} = 1 \text{mm} \]

So energy gets converted to heat in cloud at spatial scales on the order of 1 mm. Pretty small!

Typical values of $\epsilon$ range from $10^{-4}$ to $10^{-2}$ m$^2$/s$^3$ in convective clouds to $10^{-5}$ to $10^{-4}$ m$^2$/s$^3$ in stratiform clouds.

**References**
