Figure 1: A *blackbody defined by a cavity where emission and absorption are in equilibrium so as to maintain a constant temperature*

**Blackbody radiation**

The basic principles of thermal emission are as follows:

- An object of temperature $T$ radiates energy at all wavelengths (or equivalently frequencies). The amount of energy radiated at a specific frequency follows a relation known as the Planck function.

- The wavelength of peak radiation is inversely proportional to the object temperature. Equivalently, the peak frequency is proportional to the temperature.

- The total amount of energy radiated, summed over all wavelengths, is proportional to the temperature to the fourth power $F \propto T^4$

- With certain caveats (described later) an object absorbs as effectively as it emits

By definition, a blackbody absorbs all radiation incident upon it. An example is the sun. The Planck function, which describes the intensity of radiation emitted from a blackbody is (W m$^{-2}$)
The blackbody spectrum. Note that colder temperatures correspond to lower energies of emission and longer wavelengths.

\[ B_\nu (T) = \frac{2\hbar \nu^3}{c^2 (e^{\nu/kT} - 1)} \]

which, in terms of wavelength is

\[ B_\lambda (T) = \frac{2hc^2}{\lambda^5 (e^{\hbar c/\lambda kT} - 1)} \]

What is notable about this function is that it has a peak. We can derive the location of the peak (Wien’s Law) by taking the first derivative of \( B \) with respect to \( \lambda \) and setting to zero to get

\[ \lambda_{max} = \frac{2897}{T} \]

Why is there a peak? Where does this function come from? A first statement here is that the derivation of the Planck function is not straightforward, and was approached initially through combinations of classical, statistical and quantum mechanics. But it is an extremely important result, perhaps the first important result of the newly developed quantum theory. Its ingredients
have much to say about other topics in atmospheric physics. Here we will hand-wave some of the
more central concepts.

The way this problem was first approached was to think of the radiating body as a bunch of
individual oscillators. The total energy of the oscillators would be distributed among translational,
vibrational, rotational, and electronic energy

\[ e_{\text{tot}} = e_{\text{trans}} + e_{\text{rot}} + e_{\text{vib}} + e_{\text{elec}} \]

Each of these types of energy could be further subdivided into distinct independent modes, each
having total energy \( kT \), where \( k \) is the Boltzmann constant (\( 1.381 \times 10^{-23} \) J/K). Now all this
jiggling associated with the temperature corresponds to displacements of electric charge. As dis-
cussed previously this generates an induced radiation field – the dipole radiation – whose mag-
nitude varies as frequency \( \nu \) (or \( \omega \)) squared. Intuitively, the more jiggles that happen, the more
energy that is lost and the cooler the object must get. Absent external inputs, a stove doesn’t stay
hot.

In a so-called black-body, we don’t let dipole radiation just radiate away into space. Rather it is
trapped inside an enclosed “black” (non-reflecting) box. Thus, any dipole radiation that is emitted
(our \( E_0 \) in dipole radiation) and lost also acts as a source for creating more jiggles. Accordingly,
we can imagine an idealized equilibrium state in which the amount of energy that enters the system
(say through a small aperture in the box) is the same as the amount of energy that escapes it.

What is this equilibrium state? Well first, we can think of this as being like any linear system
in a condition of steady-state: the rate of change of a substance is proportional to coefficient \( a \)
times the substance (the source) minus coefficient \( b \) times the substance (the sink). In equilibrium,
\( a = b \). In our case, the energy sink is proportional to the amount of energy of the system (\( kT \)) and
to the loss rate of energy through emission (\( \propto \omega^2 \)). Contrast this with dipole radiation where the
incoming energy was external leading to a power that is proportional to \( \omega^2 \times \omega^2 \); so in a blackbody
one of the \( \omega^2 \) is now replaced by \( kT \). The final step is to determine the absorption of incoming
radiation, because the absorbed incoming radiation must be equivalent to the loss rate for there to
be an equilibrium. Things get more complicated here, but ultimately, we do not lose the central
feature that the radiant intensity is a product of the thermal energy and its frequency squared. This
“classical” approach gives us the so-called Rayleigh-Jean’s law

\[ B_{\nu,RJ} = \frac{8\pi\nu^2}{c^3}kT \]

which looks like the long tail of the Planck function. The problem here is that the Rayleigh-Jeans
law just goes to infinity for high frequencies and short wavelengths. We don’t get fried by X-rays,
Here’s the safety catch: remember, energy is “quantized”, and the amount of energy in a oscillator mode \( kT \) can never be less than \( h\nu \). When \( kT \ll h\nu \), the Rayleigh-Jeans assumption that total oscillator energy is \( kT \) is no longer correct. Now, a “quantum” description must take over. It accounts for the rapid dropoff in energy at short wavelengths, high frequencies in \( B \).

A final consideration is that in equilibrium systems follow a distribution of states in proportion to their potential energy. This is called the Boltzmann distribution, which follows

\[
  n = n_0 \exp \left( -\frac{P.E.}{kT} \right)
\]

The most familiar example of this is that in a gaseous medium (e.g. the atmosphere) density decreases with height. Making an isothermal approximation

\[
  \rho = \rho_0 \exp \left( -\frac{mgh}{kT} \right)
\]

An added ingredient for blackbody radiation is that quantum mechanical energy is quantized (i.e. \( e_n = n\hbar\nu \), where \( n = 0, 1, 2, \ldots \)). But we get something similar anyway which is that the energy distribution follows

\[
  e_{qm} = \frac{h\nu}{\exp \left( \frac{h\nu}{kT} \right) - 1}
\]

Notice that this is much like \( e^{-h\nu/kT} \) and actually would be if energies weren’t quantized. The main point here is that if energies are sufficiently high that \( h\nu \gg kT \), quantum mechanics must take over from the classical mechanics perspective. And in this perspective, radiant energy decreases with increasing frequency.

Combining the Rayleigh-Jeans and Quantum mechanical approches, we end up with our desired expression for Black-body radiation

\[
  B_\nu (T) = \frac{2h\nu^3}{c^2 \left( e^{h\nu/kT} - 1 \right)}
\]

Which implies a peak in the spectrum as prescribed by Wien’s law. So, as you can see, the derivation is involved, even in our hand-waved approach. However, it touches on a number of concepts that we will get back to repeatedly, so are worth keeping in mind.

To get the total intensity averaged over all frequencies we integrate \( B_\nu \) to get \((\text{W m}^{-2} \text{ sr}^{-1})\)

\[
  B (T) = \int_0^\infty B_\nu (T) \, d\nu = bT^4
\]
Figure 3: A planetary blackbody where emitted thermal radiation from the sphere is in balance with absorbed solar radiation.

where

\[ b = \frac{2\pi^4 k^4}{15c^2 h^3} \]

Since blackbody radiation is isotropic the blackbody flux is (W m\(^{-2}\))

\[ F_{BB}(T) = \pi b T^4 = \sigma T^4 \]

where \( \sigma = 5.67 \times 10^{-8} \text{ J m}^{-2} \text{ sec}^{-1} \text{ deg}^{-1} \) is the Stefan-Boltzmann constant.

Note that, we can show that \( F = \sigma T^4 \) using three methods

1. Stefan showed that \( F = \sigma T^4 \) based on experimental measurements
2. Boltzmann showed that \( F \propto T^4 \) based on thermodynamic arguments (for an assignment)
3. Planck showed that \( F = \sigma T^4 \), based on quantum mechanical arguments, where \( \sigma = \pi b \) is a mix of fundamental constants.

**Example**

Exercise 4.6 in Wallace and Hobbs

Calculate the equivalent blackbody temperature of the Earth, assuming a planetary albedo of 0.30. Assume that the Earth is in radiative equilibrium.

\[ F_E = \sigma T_E^4 = \frac{(1 - A) F_S}{4} = \frac{(1 - 0.30) \times 1368}{4} = 239.4 \text{ W m}^{-2} \]
Solving for $T_E$ we get

$$T_E = \left( \frac{F_E}{\sigma} \right)^{1/4} = \left( \frac{239.4}{5.67 \times 10^{-8}} \right)^{1/4} = 255 \text{ } K$$