Electromagnetic waves

An electric generator rotates a magnet to create electricity. An magnetic resonance imager found in a hospital creates a magnetic field by passing a large electric current in a circle. A changing electric field induces a magnetic field and a changing magnetic field induces an electric field. The magnetic field is in each case normal to the electric field.

The fundamental equations that describe electromagnetic radiation are Maxwell’s equations. The solutions to the equations are sinusoidal of form

\[
\vec{E} = E_0 \exp(i \vec{k} \cdot \vec{x} - i \omega t)
\]

\[
\vec{H} = H_0 \exp(i \vec{k} \cdot \vec{x} - i \omega t)
\]

where \(E\) is the electric field and \(H\) is the magnetic field, the real component of \(\vec{k}\) is the wavenumber \(2\pi/\lambda\), \(\vec{x}\) is the direction of wave propagation and \(\omega\) is the angular frequency of the radiation \(2\pi \nu\).

Some fundamental properties of the waves are that they do not diverge, that \(H\) and \(E\) are normal to each other, and that radiation of one frequency or wavelength does not interact with radiation of another frequency. The last point is particularly useful because it means we can analyse the separate contributions of a range in frequency or wavelength to the total energy input from radiation as it accumulates over time.

The symbols \(\varepsilon\) and \(\mu\) stand for the electric permittivity and magnetic permeability, respectively. They are merely properties of matter. Curiously, unlike say sound or water waves, E-M waves do not require a medium and can travel in a vacuum. The permittivity and permeability of a vacuum have the symbols \(\varepsilon_0\) and \(\mu_0\). The instantaneous flux density of the electric field in the direction of
the wave is defined by the Poynting vector

\[ \vec{S} = \vec{E} \times \vec{H} \]

In a vacuum this can be simplified to

\[ S = \frac{E_0^2}{\mu_0 c} \cos^2(\omega t) \]

Although averaged over time it is equal to the average Poynting Flux

\[ S_{av} = \frac{1}{2} \text{Re}(E_0^2 \times H_0) \]

or

\[ S_{av} = \frac{1}{2} \varepsilon_0 E_0^2 \]

\( S_{av} \) has units of \( \text{W m}^{-2} \) (or \( \text{J s}^{-1} \text{ m}^{-2} \)) and is a function of wavelength. This is units of flux (quantity per length squared per second). In atmospheric sciences, we more commonly use the symbol \( F \).

The point here is that the flux of energy \( F \) is proportional to the square of the magnitude of an electromagnetic wave. It is the flux density of electromagnetic waves that is of primary interest to climate studies.

Okay, but let’s look at interactions of radiation with something that is not a vacuum. Remember

\[ \vec{E} = \vec{E}_0 \exp \left( i \vec{k} \cdot \vec{x} - i \omega t \right) \]

Assume that the wavenumber vector \( \vec{k} \) is a complex number with a real and imaginary component, i.e.

\[ \vec{k} = \vec{k'} + i \vec{k''} \]

Therefore, the consequence of \( k \) being complex is the following

\[ \vec{E} = E_0 \exp \left( -i \vec{k''} \cdot \vec{x} \right) \exp \left( i \left( \vec{k'} \cdot \vec{x} - \omega t \right) \right) \]

Note that if \( \vec{k''} = 0 \) then the amplitude of the wave is constant.

Thus the phase of the wave is given by

\[ \phi = \vec{k'} \cdot \vec{x} - \omega t \]

At constant phase

\[ d\phi = 0 = \vec{k'} \cdot d\vec{x} - \omega dt \]
implying that the phase speed of radiation in the direction the waves propagate is given by

\[ v = \frac{d\vec{r}}{dt} = \frac{\omega}{\vec{k}} \]

or alternatively

\[ v = \nu \lambda = \frac{\omega}{2\pi} \lambda \]

The point here is that frequency and wavelength are linked through the phase speed of the wave. In a vacuum the phase speed is \( c = 3.0 \times 10^8 \) m/s, the “speed of light”, where

\[ c = \nu \lambda = \frac{\omega}{2\pi} \lambda = \frac{\omega}{k} \]

For something that is not a vacuum, we can introduce a “complex refractive index” \( N \) for describing a medium.

\[ \vec{k} = \vec{k}^' + i\vec{k}^" = \frac{\omega}{c} N = \frac{\omega}{c} (n' + in") \]

where \( N \) is the (complex) refractive index of the material with real and imaginary components \( n' \) and \( n'' \), respectively. It can be shown that the real component of the refractive index is just a measure of how much the phase speed of light is slowed by a medium

\[ n' = \frac{c}{v} \]

We showed previously that

\[ \vec{F} \propto \vec{E}^2 \]

Therefore

\[ \vec{F} = \vec{F}_0 \exp(-2\vec{k}^" \cdot \vec{x}) \]

Thus, the flux decays exponentially with distance through an absorbing medium. Now, since

\[ |\vec{k}^"| = \frac{\omega}{c} \text{Im}(N) = \frac{\omega}{c} n'' = \frac{2\pi n''}{\lambda} \]

we can define an absorption coefficient

\[ \beta_a = \frac{4\pi n''}{\lambda} \]

such that

\[ \vec{F} = \vec{F}_0 \exp(-\beta_a x) \]
This is known as Beer’s Law