The mathematics of scattering and absorption and emission

The transmittance of an layer depends on its optical depth $\tau$, which in turn depends on how much of the substance the radiation has to pass through, and how dark the substance is to the radiating wavelength. A stout (cola) transmits less light from a direct beam of radiation than a lager (gingerale), even if in the same sized glass. The direct intensity $I$ of the radiation is attenuated exponentially, so that the brightness of an object seen through the medium drops in proportion according to the most simple of first-order differential equations

$$\frac{dI}{d\tau} = -I$$

This has the standard exponential decay solution

$$I(\tau) = I(0) \exp(-\tau)$$

Thus the transmittance is

$$T = \frac{I(\tau)}{I(0)} = \exp(-\tau)$$

and the absorptance is

$$A = 1 - \exp(-\tau)$$

Note that as $\tau \to \infty$, the transmittance goes to 0 and the absorptance goes to 1.

Note that Beer’s Law expresses only extinction from a direct beam of radiation. Milk, which scatters rather than absorbs light, might have the same optical depth as a cola, and attenuate the direct radiation equally. Beer’s Law says nothing about whether a direct beam of radiation is attenuated due to scattering (as in the case of milk) or absorption (as in the case of a cola). However, the milk looks much brighter because it is scattering radiation in all directions, and this radiation becomes multiply scattered. Multiple scattering mean that the radiation is just re-directed and homogenized (made more isotropic). This is a more advanced topic.

How do we express $\tau$ in terms of physically meaningful parameters? There are a number of different ways that are used, all related. At the most fundamental level, every molecule has an effective absorption cross-section $\sigma_a$ or scattering cross-section $\sigma_s$ that has units of area. What this represents is the effective cross-sectional area of the molecule that absorbs all radiation incident upon it. For example a glass of cola is black, and probably has an absorption cross-section close to it’s actual cross-sectional area normal to the plane of incident radiation. The same glass filled with ginger-ale would have a much smaller absorption cross-section, even though the glass itself has the same physical cross-sectional area. Ginger-ale is lighter in color. The absorption and scattering cross-sections are related to the geometric cross-section $\sigma$ through a scattering or absorption
efficiency $K_{\lambda}$, which is dimensionless. Thus

$$\sigma_{s\lambda} = K_{s\lambda}\sigma$$

$$\sigma_{a\lambda} = K_{a\lambda}\sigma$$

so, for example

$$K_s = \frac{\sigma_s}{\sigma}$$

The volume absorption (or scattering) coefficient $\beta_{a}$ is the product of $\sigma_{a}$ and the concentration of molecules or particles $N$ in units of #/m$^3$

$$\beta_{a\lambda} = N\sigma_{a\lambda} = NK_{a\lambda}\sigma$$

$$\beta_{s\lambda} = N\sigma_{s\lambda} = NK_{s\lambda}\sigma$$

Thus $\beta_{a}$ has units of 1/m. Since the optical depth $\tau$ is dimensionless it follows that

$$d\tau_{a\lambda} = \beta_{a\lambda}ds = N\sigma_{a\lambda}ds = NK_{a\lambda}\sigma ds$$

$$d\tau_{s\lambda} = \beta_{s\lambda}ds = N\sigma_{s\lambda}ds = NK_{s\lambda}\sigma ds$$

where $ds$ is the pathlength of the radiation. The more soda or beer you peer through, the harder it is to see to the other side.

It is also common to express $\beta_{a}$ in terms of a mass absorption coefficient $k_{a}$ in units of m$^2$/kg such that

$$d\tau_{a\lambda} = k_{a}\rho w ds$$

where $\rho$ is the density of air and $w$ is the mixing ratio of the gas in the air. This is very close to a final, very commonly used formulation that defines the $\tau_{a}$ in terms of the mass path $u$ with units of g m$^{-2}$ such that

$$du = \rho w ds$$

and

$$d\tau_{a} = k_{a}du$$

Of course, it is important to recognize that in the atmosphere $u = u(z)$ and therefore, if the mass path is defined with respect to a vertical column.

$$\tau_{a}(z) = k_{a} \int_{z}^{\infty} \frac{du(z)}{dz} dz$$
Figure 1: Illustration of the slantpath through a vertical layer.

where \( \infty \) refers to the top of the atmosphere. Note that \( dz = \cos \theta ds \) where \( ds \) is the slant path.

Thus according to Beer’s law

\[
t_{\lambda}(z) = \exp(-k_{\lambda}u)
\]

or, coming back to the expression for exponential decay of intensity, we must modify

\[
\frac{dI_{\lambda}}{d\tau_{\lambda}} = -I_{\lambda}
\]

\[
I_{\lambda}(\tau_{\lambda}) = I_{\lambda}(0) \exp(-\tau_{\lambda})
\]

to

\[
\mu \frac{dI_{\lambda}}{d\tau_{\lambda}} = -I_{\lambda}
\]

\[
I_{\lambda}(\tau_{\lambda}) = I_{\lambda}(0) \exp(-\tau_{\lambda}/\mu)
\]

where, \( \mu = \cos \theta \), where \( \theta = 0 \) and \( \mu = 1 \) for the up direction, \( \theta = 90 \) and \( \mu = 0 \) for the
Figure 2: In this plot, the size parameter increases as one moves from the Rayleigh Scattering regime to the geometric optics regime.

side direction. And what this shows is that for a particular atmospheric optical depth, radiation is attenuated most when it is coming at side angles, because it must pass through much more atmosphere before coming out the other side than if it is passing straight through. There are a few other important “single-scattering” parameters that are necessary to describe the full radiation field:

**Size parameter**

The size parameter \( x \) is a dimensionless number that expresses the size of the droplets relative to the wavelength of incident radiation

\[
x = r |k| = \frac{2\pi r}{\lambda}
\]

where \( r \) is the particle radius, and \( \lambda \) is the wavelength of the light. Remember that we have shown that the size of a particle relative to incident radiation that is quite important in determining how light is scattered.
Radiative Transfer

to $m_{1000}/H_1$ (no absorption). For $1/H_{1088}/x/H_{1351}/50$, referred to as the Mie scattering regime, $K_{/H_9261}$ exhibits a damped oscillatory behavior, with a mean around a value of 2, and for $x/H_{1407}/50$, the range referred to as the geometric optics regime, the oscillatory behavior is less prominent and $K_{/H_9261}/H_9261/2$.

Exercise 4.9

Estimate the relative efficiencies with which red light ($/H_9261/0.64/H_9262/m$) and blue light ($/H_9261/0.47/m$) are scattered by air molecules.

Solution:

From (4.20) $I_{/H_798}/H_9261/H_20898$. Hence, the preponderance of blue in light scattered by air molecules, as evidenced by the blueness of the sky on days when the air is relatively free from aerosols. Figure 4.14 shows an example of the coloring of the sky and sunlit objects imparted by Rayleigh scattering. The photograph was taken just after sunrise. Blue sky is visible overhead, while objects in the foreground, including the aerosol layer, are illuminated by sunlight in which the shorter wavelengths (bluer colors) have been depleted by scattering along its long, oblique path through the atmosphere.

Ground-based weather radars and remote sensing of rainfall from instruments carried aboard satellites exploit the size strong dependence of scattering efficiency $K_{/H_9261}$ upon size parameter $x$ for microwave radiation in the 1- to 10-cm wavelength range incident upon clouds with droplet radii on the order of millimeters. In contrast to infrared radiation, which $K_{/H_9261}$ (blue) $K_{/H_9261}$ (red) $H_4$ ($/H_9261/3.45$), $Gustav Mie (1868–1957) German physicist. Carried out fundamental studies on the theory of electromagnetic scattering and kinetic theory.

Figure 4.12 Schematic showing the angular distribution of the radiation at visible ($0.5/H_9262/m$) wavelength scattered by spherical particles with radii of (a) 10 m, (b) 0.1 m, and (c) 1 m. The forward scattering for the 1-m aerosol is extremely large and is scaled for presentation purposes. [Adapted from K. N. Liou, An Introduction to Atmospheric Radiation, Academic Press, p. 7 (2002).]

Figure 4.13 Scattering efficiency $K_{/H_9261}$ as a function of size parameter $x$, plotted on a logarithmic scale, for four different refractive indices with $m_r/H_1.5$ and $m_i$ ranging from 0 to 1, as indicated. [From K. N. Liou, An Introduction to Atmospheric Radiation, Academic Press, p. 191 (2002).]

Figure 3: The extinction efficiency as a function of size parameter for a range of values of the complex index of refraction. More absorption brings the extinction efficiency closer to unity and a single scattering albedo of 0.5.
scattering of visible light, but they have more isotropic Rayleigh scattering of radar wavelengths. As mirrors, light is scattered over a distribution of angles. Cloud particles are strongly forward scattering. 

\[ \mu = 1 \text{ corresponds to forward scattering (the same direction as the incoming beam)}, \quad \mu = -1 \text{ corresponds to back-scattering (towards the light source)}. \]

With the exception of such things as mirrors, light is scattered over a distribution of angles. Cloud particles are strongly forward scattering of visible light, but they have more isotropic Rayleigh scattering of radar wavelengths.

**Single-scattering albedo**

The single-scattering albedo \( \omega_0 \) ranges from 0 to 1 and represents the fraction of energy incident on a particle that is scattered rather than absorbed. i.e.

\[
\omega_0 = \frac{K_s}{K_s + K_a} = \frac{\sigma_s}{\sigma_s + \sigma_a} = \frac{\tau_s}{\tau_s + \tau_a}
\]

In the visible, clouds and Rayleigh scattering gases like N\(_2\) and O\(_2\) have a single scattering albedo unity (i.e. no absorption). In their IR absorption bands, gases have a single-scattering albedo of zero (i.e no scattering).

**Phase function**

If light is scattered \( (\omega_0 > 1) \), the probability it is scattered in any given direction is expressed by the phase function \( p(\mu) \), where \( \mu = \cos \theta \), where \( \theta \) is relative to the direction of the incoming beam. \( \mu = 1 \) corresponds to forward scattering (the same direction as the incoming beam), and \( \mu = -1 \) corresponds to back-scattering (towards the light source). With the exception of such things as mirrors, light is scattered over a distribution of angles. Cloud particles are strongly forward scattering of visible light, but they have more isotropic Rayleigh scattering of radar wavelengths.

Figure 4: A graphical representation of the phase function with forward scattering at zero degrees to the right for a) Rayleigh Scattering b) Mie Scattering and c) the Geometric Optics regime. Note that the larger the size parameter, the more strongly forward scattering is the particle.
Gases have isotropic Rayleigh scattering in the visible also.

In general $p(\mu)$ is normalized such that

$$\frac{1}{2} \int_{-1}^{1} p(\mu) \, d\mu = 1$$