1. Problem 1.1

The spectral number density expressed as a function of frequency is

\[ n(\omega) = \frac{P_e(\omega)}{c\hbar\omega} \]

whereas expressed as a function of wavelength is

\[ n(\lambda) = \frac{P_e(\lambda)}{hc^2/\lambda} \]

From the corresponding Planck functions we have

\[ n(\omega) = \frac{\omega^2}{4\pi^2 c^3} \frac{1}{\exp \left( \frac{\hbar\omega}{k_B T} \right) - 1} \]

\[ n(\lambda) = \frac{2\pi}{\lambda^4} \frac{1}{\exp \left( \frac{hc}{\lambda k_B T} \right) - 1} \]

Find the maxima of both functions by setting the derivative equal to zero. This yields the transcendental equation for the maximum of \( n(\omega) \)

\[ xe^x = 2(e^x - 1) \]

the solution to which is \( x = 1.5936 \). The equation for the maximum of \( n(\lambda) \) is obtained from

\[ xe^x = 4(e^x - 1) \]

the solution to which is \( x = 3.9207 \). The corresponding displacement laws are

\[ \lambda_m T = 9042 \mu m K \]
and
\[
\lambda_m T = 3675 \mu m K
\]

2. Problem 1.23ff

Energy has the dimensions of momentum times speed or, stated another way, momentum has the dimensions of energy divided by speed. The energy of a photon is \( h\nu \). To obtain from this energy a quantity with the dimensions of momentum requires dividing by a quantity with dimensions of speed. The only speed that seems relevant is the free-space speed of light, \( c \). Thus the momentum of a photon should be proportional to \( h\nu/c \). By dimensional analysis we cannot determine the proportionality factor, which happens to be 1.

Angular momentum has the dimensions of momentum times length, which from the above would lead to a dimensional argument for \( h \) being the angular momentum. In reality it is \( \hbar = h/(2\pi) \).

3. Problem 1.32

Momentum flux (spectral) is given by \( n(\omega) c\hbar\omega/c = n(\omega) h\omega \), where \( n(\omega) \) is the number of photons per unit frequency interval, all taken to be propagating in the same direction. The flux of any property is the number density of whatever carries that property, their speed, and the property (mass, momentum, charge, etc.). The total momentum flux is therefore

\[
\int n(\omega) \hbar\omega d\omega
\]

where integration is over the solar spectrum. The corresponding irradiance is

\[
F = \int n(\omega) c\hbar\omega d\omega
\]

and the momentum flux is therefore \( F/c \), which we might have guessed given that the di-
mensions of this quantity are the same as momentum flux, which on physical grounds must be proportional to irradiance. Momentum flux has the dimensions of pressure. A beam that illuminates a black object transfers all its momentum to it. A beam that illuminates an object with reflectivity 1 changes its momentum by twice this amount. Recall that force is rate of change of momentum and that momentum is a vector.

The radiation pressure for an irradiance of 1000 W/m² is $3.3 \times 10^{-6}$ Pa. Sea-level atmosphere pressure is about $10^5$ Pa, and so the radiation pressure of solar radiation illuminating a black object is $3 \times 10^{-11}$ times sea-level atmospheric pressure.

4. Problem 1.36

As a guess, plants use the shortwave part of the solar spectrum for photosynthesis whereas the longwave part would serve only to heat them. Absorption is greatest where it is most needed and least where it is not needed or might even be harmful to the plant.

5. Problem 1.29 (Bonus for the second part)

This follows from the theorem for change of variable of integration. We have

$$\int_{\omega_1}^{\omega_2} P_{\epsilon} (\omega) \, d\omega = \int_{x(\omega_1)}^{x(\omega_2)} P_{\epsilon} (x) \frac{d\omega}{dx} \, dx$$

where $\omega (x)$ is any transformation of variable and $x (\omega)$ is its inverse (assumed to be single-valued). We want the frequency at which the integrals are half their total value, $\sigma T^4$. That is

$$\int_{\omega_1}^{\omega_{med}} P_{\epsilon} (\omega) \, d\omega = \int_{x(\omega_1)}^{x(\omega_{med})} P_{\epsilon} (x) \frac{d\omega}{dx} \, dx$$

Thus, if we know the median frequency we can find the corresponding median $x$ (e.g. wavelength) from the inversed transformation $x (\omega)$. To find the median frequency as a function of temperature (displacement law) we need to solve the equation

$$\int_{\omega_1}^{\omega_{med}} P_{\epsilon} (\omega) \, d\omega = \sigma T^4$$
which requires numerical integration. From Eq. (1.27) it follows that the median frequency is

$$\frac{k_B T^4}{4\pi^2 c^2 \hbar^3} \int_0^{x_{med}} \frac{x^3}{\exp(x) - 1} \, dx = \frac{1}{2} \sigma T^4$$

where \( x = \hbar \omega / k_B T \). From Eq. (1.28)

$$\sigma = \frac{k_B^4}{4\pi^2 c^2 \hbar^3} \int_0^{x_{med}} \frac{x^3}{\exp(x) - 1} \, dx$$

and so we have to solve

$$\int_0^{x_{med}} \frac{x^3}{\exp(x) - 1} \, dx = \frac{1}{2} \int_0^\infty \frac{x^3}{\exp(x) - 1} \, dx$$

The resulting displacement law is

$$\lambda_{med} T = 4110 \mu m K s$$

6. Treating a photon gas as an ideal gas, show that the relationship between its pressure and its internal energy is \( p = u/3 \). To do this you will need to know that the effective “mass” of a photon can be determined by relating \( E = mc^2 \) to \( E = h\nu \), and then drawing an analogy to an ideal molecular gas that

$$p = \frac{1}{3} n m \langle v^2 \rangle$$

A photon has no rest mass. Its mass is due entirely to its speed, so that \( mc^2 = h\nu \). Substituting into the the expression for \( p \) and taking the RMS velocity as \( c^2 \) we get

$$p = \frac{1}{3} n h\nu = u/3$$
7. Express the first law of thermodynamics

\[ dh = du + dw \]

for a photon gas. Keep \( dh \) on the LHS.

*Here the work density is the change in pressure, so*

\[ dh = du + dp = nh\nu + \frac{1}{3}nh\nu = \frac{4}{3}nh\nu \]

*Alternatively, taking \( u = bT^4 \) and \( p = bT^4/3 \) where \( b \) is a constant*

\[ dh = 4bT^3dT + \frac{4}{3}bT^4dV \]

8. From 7. suppose a radiating blackbody (say a cloud) increases its volume while maintaining constant atmospheric pressure, for example by spreading like a cirrus anvil. How much energy would need to be added to the cloud for its temperature to stay constant (in which case it neither rises or sinks). Assume that the cloud is in local thermodynamic equilibrium with the atmosphere such that the radiative temperature and the temperature of molecular motions are the same. Can you imagine a situation involving a cloud where constant temperature might be possible?

*Energy loss from an anvil with volume \( V = A\delta z \) where \( A \) is the area and \( \delta z \) is the constant thickness increases at rate \( pd\ln V \) assuming temperature is constant. Thus heating must balance \( pd\ln V \) work. In other words*

\[ \frac{dh}{dt} = p\frac{d\ln V}{dt} = p\frac{d\ln A}{dt} = \frac{b}{3}T^4\frac{d\ln A}{dt} \]