1. W&H 4.12

Remote sensing in the microwave part of the spectrum relies on radiation emitted by oxygen molecules at frequencies near 55 GHz. Calculate the wavelength and wavenumber of this radiation.

From (4.2)

\[ \lambda = \frac{c^*}{\nu} = \frac{3 \times 10^8 \text{ m s}^{-1}}{55 \times 10^9 \text{ s}^{-1}} = 5450 \mu\text{m} \]

2. The flux or power of radiated light is proportional to the electric field squared, i.e. \( F \propto E^2 \)

where

\[ E = E_0 \exp \left( i \left( \vec{k} \cdot \vec{x} - \omega t \right) \right) \]

If there is absorption of this radiation in a medium, then the wavenumber \( k = k' + ik'' \) is complex. Show that in this case the time averaged flux is given by

\[ \bar{F} = \bar{F}_0 \exp \left( -2k'' \cdot \vec{x} \right) \]

where the averaging is done over a time period much longer than the period of the oscillation. What must be the sign of \( k'' \)? Explain in words, and with a drawing, what \( k'' > 0 \) implies for what happens to the flux of radiation as it propagates through an absorbing medium.

*From the complex form of the wavenumber \( \vec{k} \)*

\[ \vec{k} = \vec{k}' + ik'' \]

Therefore

\[ ik \cdot \vec{x} = i\vec{k}' \cdot \vec{x} - \vec{k}'' \cdot \vec{x} \]

and substituting

\[ \bar{E} = E_0 \exp \left( -2\vec{k}'' \cdot \vec{x} \right) \exp \left( i \left( \vec{k}' \cdot \vec{x} - \omega t \right) \right) \]

Since the flux is proportional to the electromagnetic field squared:

\[ F \propto E^2 = E_0 \exp \left( -2\vec{k}'' \cdot \vec{x} \right) \exp \left( 2i \left( \vec{k}' \cdot \vec{x} - \omega t \right) \right) \]

If this equation is time averaged, then the temporal component of the oscillatory signal is not resolved and it can be treated as a being a constant (the RMS value)

\[ F \propto E^2_0 \exp (-2k''x) \]

or

\[ F = F_0 \exp (-2k''x) \]

Since the flux is decaying, this must be due to absorption. The plot for flux should be one of exponential decay with flux on the y axis and distance on the x-axis.
3. The complex refractive index of a substance is given by \( n = n' + in'' \). The relationship of \( n'' \) to \( k'' \) is
\[
k'' = \frac{\omega}{c} n'' = \frac{2n''}{\lambda}
\]
For the case described above that the \textit{time averaged} flux is given by
\[
\vec{F} = \vec{F}_0 \exp \left(-2k'' \cdot \vec{x}\right)
\]
argue that the depth that radiation penetrates into a substance is
\[
\delta = \frac{\lambda}{4\pi n''}
\]
Substituting
\[
k'' = \frac{2n''}{\lambda}
\]
into
\[
\vec{F} = \vec{F}_0 \exp \left(-2k'' \cdot \vec{x}\right)
\]
we get
\[
\vec{F} = \vec{F}_0 \exp \left(-\frac{4\pi n'' x}{\lambda}\right)
\]
The depth \( x = \delta \) at which radiation fall to \( 1/e \) of its value at entering the substance is thus where
\[
4\pi n'' \delta / \lambda = 1
\]
Thus
\[
\delta = \frac{\lambda}{4\pi n''}
\]

4. Satellites use infrared wavelengths of \( \sim 10 \, \mu m \) to detect the surface temperature of the Great Salt Lake. The complex component of the refractive index at these wavelengths is \( \sim 0.1 \). With reference to your answer to the question above, what temperature exactly are the satellites “seeing”?

Substituting the values into the equation
\[
\delta = \frac{\lambda}{4\pi n''}
\]
we find that \( \delta \approx 10 \, \mu m \). Thus satellites that use IR to sense the temperature of the Great Salt Lake are only “seeing” the very surface.

5. Based on the fact that the diameter of the sun’s disk is 0.5°, show that the solid angle subtended by the sun is \( 6 \times 10^{-5} \text{sr} \). Hint: apply the small angle approximation \( \sin \theta \sim \theta \) when calculating \( \Delta \omega \)

The trick here is to recognize that we are talking about small angles for the purpose of calculating solid angles. Therefore, from the Taylor series expansion of \( \sin \theta \), taken to first order,
\[
\delta \omega = \sin \theta d\theta d\phi \simeq \theta d\theta d\phi
\]
\[ \Delta \omega \approx \frac{1}{2} \left( \frac{\theta}{2} \right)^2 \Delta \phi \]

where \( \theta/2 \) is the angular radius, and \( \Delta \phi = 2\pi \). Therefore

\[ \Delta \omega = \frac{2\pi}{2} (0.5 \times \pi/360)^2 = 6 \times 10^{-5} \text{ sr} \]

6. Show that for isotropic radiation (i.e., the intensity of radiation is constant in all directions) that the hemispheric flux can be described by

\[ F = \pi I \]

This can be a useful identity to remember. (Hint: use the transformation \( \mu = \cos \theta \) so that \( F = \int_0^{\pi/2} \int_0^{2\pi} I \mu d\mu d\phi \))

The trick here is to recognize that \( \int_0^{\pi/2} I \cos \theta \sin \theta d\theta \) is equivalent to \( \int_0^1 I \mu d\mu \), where \( \mu = \cos \theta \). Or you can just look up the integral in a book, but in radiative transfer, this is the way things are usually done, by doing the transformation \( \mu = \cos \theta \).

\[ F = \int_0^{2\pi} \int_0^1 I \mu d\mu d\phi = 2\pi \left(1/2\mu^2\right) \big|_0^1 = \pi I \]

7. The total solar radiation flux incident on a surface due to wavelengths between 0.3 \( \mu \text{m} \) and 1.0 \( \mu \text{m} \) is 200 W m\(^{-2}\)

   (a) what is the average spectral flux within this interval? Give your answer in units of W m\(^{-2}\) \( \mu \text{m}^{-1} \)

   (b) If the spectral flux is constant with wavelength, then what is the total flux contributed by wavelengths just between 0.4 \( \mu \text{m} \) and 0.5 \( \mu \text{m} \)?

   (c) What is the total flux (in W m\(^{-2}\)) contributed by radiation of exactly 0.5 \( \mu \text{m} \) wavelength?

\[ \text{Since the total flux is } F = 200 \text{ W m}^{-2}, \text{ then } F_\lambda = \Delta F/\Delta \lambda = 200/0.7 = 285.7 \text{ W m}^{-2} \text{ \( \mu \text{m}^{-1} \). The fraction contained in 0.1 \( \mu \text{m} \) of this is then } 285.7 \times 0.1 = 28.6 \text{ W m}^{-2}. \text{ At exactly 0.5 \( \mu \text{m} \), there is no energy, because the wavelength window is infinitely narrow and } \Delta \lambda = 0: 285.7 \times 0 \mu \text{m} = 0 \text{ W m}^{-2}. \]

8. Show that the average intensity of the Sun’s disk as seen from a distance \( D \) from the Sun’s center is given by \( P/(4\pi^2 R^2) \), where the total radiant power output is \( P \) (joules per second) and the solar radius is \( R \). For simplicity, assume that the Sun’s intensity \( I \) is the same at all points on the visible disk. You will need to derive an exact expression for the solid angle subtended by the Sun’s disk for \( D \gg R \). Note that your final solution for \( I \) should not depend on \( D \!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!"
The solid angle of the sun is given by

$$\delta \omega = \sigma / D^2 = \pi R^2 / D^2$$

The intensity of the sun is given by the ratio of the flux to the solid angle, therefore, through substitution of the above two equations:

$$I = F / \delta \omega = P / \left( 4\pi R^2 \right)$$

Thus, the intensity is independent of distance $D$. Energy is conserved, because although energy per solid angle is constant, the sun itself occupies a smaller solid angle in the sky as distance increases: $\omega \propto D^{-2}$.