Assignment 2: Due January 26, 2018

1. W&H 4.13. What you are doing here is calculating the “Jacobian”

The spectrum of monochromatic intensity can be defined either in terms of wavelength $\lambda$ or wavenumber $\nu$ such that the area under the spectrum, plotted as a linear function of $\lambda$ or $\nu$ is proportional to intensity. Show that $I_\nu = \lambda^2 I_\lambda$.

$$dI = I_\lambda d\lambda = I_\nu d\nu$$

From (4.1)
$$\nu = \frac{1}{\lambda}$$

from which it follows that
$$d\nu = -\frac{d\lambda}{\lambda^2}$$

Substituting for $d\lambda$ in the first expression, cancelling the common factor $d\nu$, and ignoring the minus sign, which is taken into account by reversing the direction of the integration, we obtain

$$I_\nu = \lambda^2 I_\lambda$$

2. Building on the question above, prove to yourself that the “peak” in the blackbody spectrum for $B_\nu$ is not the same as the peak for $B_\lambda$. You do not need to calculate $dB_\nu/d\nu$.

From the prior question
$$I_\nu = \lambda^2 I_\lambda$$

so
$$B_\nu = \lambda^2 B_\lambda$$

implying that
$$B_\nu (\nu_{\text{max}}) \neq B_\lambda (\lambda_{\text{max}})$$

3. Taking the equation for the Planck function

$$B_\nu (T) = \frac{2h\nu^3}{c^2 (e^{h\nu/kT} - 1)}$$

Prove to yourself that you obtain the classical Rayleigh-Jeans expression in the low frequency limit where $h\nu \ll kT$. You will want to make a Taylor series expansion of $e^x$ where $x = h\nu/kT$
To show this you need to invoke the Taylor series expansion of \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \) and truncate the expansion to the first order terms for which \( x \) is not negligibly small. \( x \) here is \( h\nu/kT \).

4. Classical theory tells us that each degree of freedom associated with an oscillator in a black-body should have energy \( E = kT \). Quantum theory tells us this can be wrong if it requires that \( kT \) be smaller than the universal quantum of energy \( E = h\nu \). Classical theory also tells us that the energy emitted by a blackbody increases exponentially with frequency according to the Rayleigh-Jeans law. Instead we measure, and can derive from quantum theory, the Planck function: the Planck function has the long tail of the Rayleigh-Jeans law at low frequencies, but unlike the Rayleigh-Jeans law, a sudden drop-off at high frequencies. Show that the wavelength where this drop-off starts to occur (i.e. near where there is a maximum in the Planck function) is roughly where classical theory would (impossibly) require \( kT \ll h\nu \). It may help to consider Wien’s law and look at a Planck function curve for guidance. Consider two cases, the temperature of the sun (\( T = 6000 \) K), and the temperature of the earth (\( T = 300 \) K). Keep in mind you aren’t required to explicitly solve the Planck function. Your answer shouldn’t be very long.

Where does the drop-off occur? From Wien’s Law \( \lambda_{\text{max}} = \frac{5107}{T} \) which leads to a peak wavelength for the maximum of \( B_\nu \) (not \( B_\lambda \)). Thus for Earth as an example, \( \lambda_{\text{max}} = 17 \) \( \mu \)m. The corresponding frequency is \( \nu_{\text{max}} = \frac{c}{\lambda_{\text{max}}} \) which is (taking care of the units!) equal to \( 1.8 \times 10^{13} \) Hz. In this case, we find that \( h\nu_{\text{max}} = 1.2 \times 10^{-20} \) Joules and that \( kT \) is \( 4.1 \times 10^{-21} \) Joules. This is impossible! \( h\nu \) is the smallest possible unit of energy so it cannot be greater than the average energy \( kT \). Still, they are roughly the same order of magnitude, and it is around this frequency that the drop-off in intensity occurs. The Planck equation is different than the Rayleigh-Jeans law precisely because \( kT \) cannot be less than \( h\nu \).

5. Show that the equilibrium temperature of the surface of the moon is 273 K assuming it has an albedo of 0.08.

We start with the radiative balance equation discussed in class

\[
4\pi R_m^2 \sigma T^4 = \pi R_m^2 (1 - A) S
\]

The value of \( S \) is the same for the moon as the earth, give or take. We don’t need to know \( R_m \) since the terms cancel. The answer should be 273K

6. W&H 4.21a and b. Hint: this is easiest if you use logarithms. For numbers in the seasonal variations, the eccentricity of the Earth’s orbit is \( \pm 1.7\% \) in which case you should obtain for 4.21a a variation in \( T_E \) of \( \pm 2.2 \) K and 0.91K for 4.21b.
Show that for small variations in the earth’s radiation balance
\[
\frac{\delta T_E}{T_E} = \frac{1}{4} \frac{\delta F_E}{F_E}
\]
where \( T_E \) is the planet’s equivalent blackbody temperature and \( F_E \) is the flux of radiation emitted from the top of its atmosphere. Use this relationship to estimate the change in effective temperature that would occur in response to (a) the seasonal variations in the sun-earth distance due to the eccentricity of the earth’s orbit (presently \( \sim 3\% \)), (b) an increase in the earth’s albedo from 0.30 to 0.31.

From (4.12)
\[
F = \sigma T^4
\]
Taking the log yields
\[
\ln F = 4 \ln T
\]
Taking the differential yields
\[
\frac{\delta F}{F} = 4 \frac{\delta T}{T}
\]
and dividing both sides by 4 yields
\[
\frac{\delta T_E}{T_E} = \frac{1}{4} \frac{\delta F_E}{F_E}
\]  \hspace{1cm} (1)

(a) From the inverse square law
\[
F_E = \text{const} \times d^{-2}
\]
where \( d \) is the earth-sun distance. Taking the log yields
\[
\ln F_E = \ln \text{const} - 2 \ln d
\]
Taking the differential yields
\[
\frac{\delta F_E}{F_E} = -2 \frac{\delta d}{d}
\]  \hspace{1cm} (2)

Combining (1) and (2) yields
\[
\frac{\delta T_E}{T_E} = -\frac{1}{2} \frac{\delta d}{d}
\]  \hspace{1cm} (1)
Substituting \( \delta d/d = 0.03 \) yields \( \delta T_E/T_E = -0.06 \). If \( T_E = 255 \) K, then \( \delta T_E = 1.5 \) K.
(b) \[ F_E = \text{const} \times (1 - A) \]

where \( A \) is the planetary albedo. Taking the log yields

\[ \ln F_E = \ln \text{const} - \ln(1 - A) \]

Taking the differential yields

\[ \frac{\delta F_E}{F_E} = \frac{\delta(1 - A)}{(1 - A)} \]  \hspace{1cm} (2)

Combining (1) and (2) yields

\[ \frac{\delta T_E}{T_E} = \frac{1}{4} \frac{\delta(1 - A)}{(1 - A)} \]  \hspace{1cm} (3)

Substituting \( \delta d/d = 0.01/0.70 \) yields \( \delta T_E/T_E = -0.00357 \). If \( T_E = 255 \text{ K} \), then \( \delta T_E = 0.91 \text{ K} \).