Once droplets are activated, how do they grow? There are two primary components to this problem. Growth by vapor diffusion, and growth by collision-coalescence.

**Vapor Diffusion**

Let’s say we have a droplet of radius $r$ in a supersaturation environment. Immediately adjacent to the surface of the droplet the concentration of water vapor molecules is $n_r$, and far from the droplet the concentration is $n_∞$. The rate of growth of the droplet depends on two things

1. The gradient in vapor concentration from the surface of the droplet to the ambient environment.
2. The surface area $4πr^2$

We solve the problem of droplet growth using the Diffusion Equation. This is the same equation used for describing heat transfer.

$$\frac{dn}{dt} = D∇^2n$$

where $D$ is the diffusivity of water vapor molecules (like a flux: units $cm^2/s$).

If we assume the ambient vapor field is in steady-state, which is not true, but a perfectly justifiable assumption for cloud processes.

$$\frac{dn}{dt} = 0$$

then

$$∇^2n = 0$$

If we assign $x$ as the radial dimension, in spherical coordinates, we get

$$\frac{∂^2n}{∂x^2} + \frac{1}{2x} \frac{∂n}{∂x} = 0$$

which has the general solution

$$n(x) = C_1 - \frac{C_2}{x}$$

Applying the boundary conditions $x → ∞, n → n_∞; x → r, n → n_r$, we find the vapor field can be described by

$$n(x) = n_∞ - \frac{r}{x} (n_∞ - n_r)$$

The flux of water vapor molecules to the surface of the droplet is described by

$$\vec{j} = D \frac{∂n}{∂x} \bigg|_{x=r}$$

which has units of #/$m^2$s. The number increase $N$ in condensed molecules is also proportional to area

$$\frac{dN}{dt} = 4πr^2D \frac{∂n}{∂x} \bigg|_{x=r}$$
If $m_w$ is the mass of one water molecule

\[
\frac{dm}{dt} = 4\pi r^2 D \frac{\partial n}{\partial x} \bigg|_{x=r} m_w
\]

\[
= 4\pi r D (n_\infty - n_r) m_w
\]

\[
\frac{dm}{dt} = 4\pi r D (\rho_\infty - \rho_r)
\]

So rate of mass accumulation of droplets is proportional to their size. Note that the above equation is for a single droplet only, and that we would need to multiply the above equation by the droplet number concentration to get total mass. What stops droplets from growing indefinitely? The above equation is based on the faulty assumption that the vapor field does not change. Naturally, as droplets grown, they will deplete the available water from their surroundings, thereby reducing the vapor concentration gradient between the ambient air and the surface of the droplet.

How fast do droplets grow in size? Starting from $m = \frac{4}{3} \pi \rho_l r^3$

\[
\frac{dm}{dt} = 4\pi \rho_l r^2 \frac{dr}{dt}
\]

\[
\frac{dr}{dt} = \frac{D}{\rho_l} (\rho_\infty - \rho_r)
\]

Using the ideal gas law for water vapor

\[
\frac{dr}{dt} = \frac{D}{r \rho_l R_v T} (e_\infty - e_r)
\]

\[
\frac{dr}{dt} = \frac{D \rho_v e_\infty}{r \rho_l e_\infty} (e_\infty - e_r)
\]

which, assuming $e_{s\infty} \simeq e_s$, we can show that

\[
\frac{e_\infty - e_r}{e_\infty} \simeq \frac{e_\infty - e_s}{e_s} \equiv S - 1 = s
\]

where $S$ is the saturation ratio and $s$ is the supersaturation. So

\[
r \frac{dr}{dt} = G_1 s
\]

where

\[
G_1 = \frac{D \rho_v e_\infty}{\rho_l} = \frac{D e_s(T)}{\rho_l R_v T}
\]

We have made a number of assumptions in these derivations that we should be aware of.

1. All water vapor molecules that hit the droplet stick. The reality may be that only 3% do. This matters mostly during initial growth of the droplet.

2. The temperature at the surface of the droplet is the same as the ambient air. In reality it is warmer due to the latent heat associated with condensation. This retards growth
3. The droplet is stationary with respect to its environment. A droplet that falls as it grows is “ventilated” which changes the distribution of water vapor around the droplet and carries away latent heat. Apparently, however, this effect is very small, and can usually be neglected.

If we integrate the growth rate equation we get

\[ r(t) = \sqrt{r_0^2 + 2Gst} \]

where \( r_0 = r^* \), the activation radius if we are starting from a haze particle. A table of the amount of time it takes for a particle to reach a given size is given below. Note that droplets grow very rapidly at first, and as the droplets grow their initial size matters less and less.

Figure 1: From Rogers and Yau (p103) The assumed supersaturation is 0.05%, \( p = 900 \text{ mb} \) and \( T = 273 \text{ K} \).

It is easy to see from this table why the edges of clouds are so sharp. Only falling drops larger than about 0.1 mm will give any blurring to the cloud, and droplets must be significantly bigger in order to have any chance of hitting the ground, particularly if cloud base if very high (as it often is here in Utah in Summer).

It is important to recognize that when a water drop evaporates, it does not completely disappear, but rather becomes a haze particle whose size is determined by the Kohler equation according to the mass of solute it contains, and the ambient relative humidity (aka saturation ratio).

<table>
<thead>
<tr>
<th>Radius (( \mu m ))</th>
<th>Time (sec) to grow from initial radius 0.75 ( \mu m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4, 0.15, 0.013</td>
</tr>
<tr>
<td>2</td>
<td>130, 7.0, 0.61</td>
</tr>
<tr>
<td>4</td>
<td>1,000, 320, 62</td>
</tr>
<tr>
<td>10</td>
<td>2,700, 1,800, 870</td>
</tr>
<tr>
<td>20</td>
<td>8,500, 7,400, 5,900</td>
</tr>
<tr>
<td>30</td>
<td>17,500, 16,000, 14,500</td>
</tr>
<tr>
<td>50</td>
<td>44,500, 43,500, 41,500</td>
</tr>
</tbody>
</table>

TABLE 7.2. Rate of Growth of Droplets by Condensation (initial radius 0.75 \( \mu m \)). (From Mason, 1971)
### TABLE 7.3. Distance a Drop Falls before Evaporating, assuming Isothermal Atmosphere with $T = 280\, K$, $S = 0.8$

<table>
<thead>
<tr>
<th>Initial radius</th>
<th>Distance fallen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \mu m$</td>
<td>$2 \mu m$</td>
</tr>
<tr>
<td>$3 \mu m$</td>
<td>$0.17, mm$</td>
</tr>
<tr>
<td>$10 \mu m$</td>
<td>$2.1, cm$</td>
</tr>
<tr>
<td>$30 \mu m$</td>
<td>$1.69, m$</td>
</tr>
<tr>
<td>$0.1, mm$</td>
<td>$208, m$</td>
</tr>
<tr>
<td>$0.15 , mm$</td>
<td>$1.05, km$</td>
</tr>
</tbody>
</table>

Figure 2: From Rogers and Yau (p105)