How do we get rain? So far we’ve discussed droplet growth by vapor diffusion, but this is not the process that by itself is primarily responsible for precipitation in warm clouds. The primary production mechanism for growth of precipitation sized droplets is “collision-coalescence”. In this process droplets collide due to differential settling velocities, and coalesce. If we say the big collector drop is $r_1$ and the smaller drops it is collecting have radius $r_2$ and $r_1 \gg r_2$ then the rate of collection depends on

1. The collision cross-sectional area is
   \[ \pi (r_1 + r_2)^2 \simeq \pi r_1^2 \]

2. The relative collection velocity is
   \[ v_{T1} - v_{T2} \simeq v_{T1} \]

3. The coalescence efficiency is approximately unity if $r_1 > 10r_2$ (Fig 6.22 in W&H)

4. The collision efficiency $E(r_1, r_2)$ is given by figure 6.20 in W&H

5. The LWC of the collected droplets, represented in W&H by the symbol $w_l$

Therefore the growth rate of the collector drop is

\[ \frac{dm_1}{dt} = \pi r_1^2 E(r_1, r_2) v_{T1} w_l \]

Noting that

\[ \frac{dm}{dt} = \frac{dm}{dr} \frac{dr}{dt} \]

and

\[ \frac{dm}{dr} = 4\pi \rho r^2 \]

we can show that (W&H 6.28)

\[ \frac{dr_1}{dt} = \frac{v_{T1} w_l E(r_1, r_2)}{4\rho_l} \]

Note that $r_1 > 20 \mu m$ appears to be a critical initial size for initiation of the collision-coalescence process. If droplet are unable to grow to this size in sufficient concentrations by vapor diffusion, then typically rain does not form.

**Terminal velocity**

In the “Stokes regime”, the terminal velocity is a balance between a drag force and the gravitational force

\[ 6\pi \eta v_T r = \frac{4}{3} \pi \rho r^3 g \]

where $\eta$ is the
\[ v_T = \frac{2\rho r^2 g}{9\eta} = k_1 r^2 \]

where \( k_1 \simeq 1.19 \times 10^6 \text{ cm}^2 \text{s}^{-1} \). However this assumes “Stokes” drag. However, for rain drops with radius \( 0.6 \text{ mm} < r < 2 \text{ mm} \) the drag force is in fact proportional to \( r^2 \). The solution then for \( v_T \) is

\[ v_T \simeq k_2 r^{1/2} \]

where \( k_2 \simeq 2.01 \times 10^3 \text{ cm}^{-1} \text{s}^{-1} \). In the intermediate range \( 40 \mu \text{m} < r < 0.6 \text{ mm} \)

\[ v_T \simeq k_3 r \]

where \( k_3 \simeq 8 \times 10^3 \text{ s}^{-1} \)

Terminal velocities as a function of droplet size are given in the table below.

<table>
<thead>
<tr>
<th>Diam. (mm)</th>
<th>Fall speed (m/s)</th>
<th>Diam. (mm)</th>
<th>Fall speed (m/s)</th>
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<tr>
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<td>7.27</td>
<td>5.8</td>
<td>9.17</td>
</tr>
</tbody>
</table>

Figure 1: Table of terminal fall speed from Rogers and Yau Table 8.1

**Collision Efficiency**

The computation of the collision efficiency from first principles is a very difficult problem owing to the complexities of the flow field around a sphere falling through a viscous medium. Rogers and Yau state “For any size of collector drop, the collision efficiency is small for small values of \( r_2/r_1 \). The collected droplets are then small, have little inertia, and are easily deflected by the flow around the collector drop. The inertia of the droplets increases with \( r_2/r_1 \) accounting for an increase in
collision efficiency up to a radius ratio of about 0.6. Two counteracting effects come into play as $r_2/r_1$ increases beyond this value. Because the difference in the size of the drops is getting smaller, the relative velocity between the drops is reduced, prolonging the time of interaction. The flow fields interact strongly, and the time can be sufficient for the droplet to be deflected around the drop without collision. On the other hand there is a possibility for a trailing droplet to be attracted into the wake of a drop falling close by at nearly the same speed. This effect can lead to “wake capture” and to collision efficiencies that exceed unity for values of $r_2/r_1 \approx 1$.

There is also a coalescence efficiency, as shown by Fig 6.22 in Wallace and Hobbs. Big drops can bounce of each other. Really, though it is the collision efficiency that is of most interest for collision-coalescence, because the concentrations of these bouncy droplets are so low.

Figure 2: From Rogers and Yau
Example

Imagine a cumulus cloud with and LWC of 0.4 g m$^{-3}$ 1 km thick with a monodisperse droplet mode at 10 $\mu m$ radius and a second mode with concentrations of 1/litre at 100 $\mu m$ radius at cloud top. The updraft velocity $w$ in the cloud is 0.05 m/s.

1. Derive an expression for the size of the drizzle drops as a function of distance as they descend through the cloud

$$\frac{dr}{dt} = \frac{w_1 v_T E (r_1, r_2)}{4 \rho_l}$$

This distance the droplet falls from cloud top is $h = v_T t$, so

$$\frac{dr}{dt} \frac{dh}{dt} = \frac{dr}{dh} = \frac{LWC v_T E (r_1, r_2)}{(w - v_T) 4 \rho_l}$$

Thus,

$$\int_0^H w_l dh = 4 \rho_l \int_{r_0}^{r_H} \frac{w - v_T}{v_T E} dr_1$$

Okay. This is a bit nasty, but it can be solved for $r_1 (H)$ if we know $v_T (r)$ and we assume $w_1 \neq w_1 (h)$.

$$w_1 H = 4 \rho_l \left[ \int_{r_0}^{r_H} \frac{w}{v_T (r_1) E} dr_1 + \int_{r_0}^{r_H} \frac{1}{E} dr_1 \right]$$

If we assume that in general $w \ll v_T$ (which maybe isn’t so good), integrating we get the radius in meters at cloud base as

$$r = r_0 + \frac{w_1}{4 \rho_l} h = 100 \times 10^{-6} + 8.1 \times 10^{-8} H$$

2. What is the droplet size at cloud base?

Just substitute $h = 1000 m$ to get $r = 180 \mu m$

3. What is the precipitation flux in mm/hr

The drizzle drops are in the intermediate size range for falls speed $v_T = k_3 r$, so

$$Flux = LWC_{precip} \times v_{Tprecip}$$

$$Flux = \frac{4}{3 \pi} \rho r_1^3 N \times k_3 r$$

$$Flux = \frac{4}{3 \pi} \left(1000\right) \left(180 \times 10^{-6}\right)^3 \left(1 \times 10^4\right) 8 \times 10^3 \left(180 \times 10^{-6}\right)$$

with appropriate conversions (divide by $\rho_l = 1000$ and multiply by the 86400 s in a day and 1000 mm per m), I get 0.4 mm/day.

4. In the absence of updrafts generating new cloud, what is the time scale for depletion of all the liquid water in the cloud?

$$LWC = LWC_0 \exp (-\varphi t)$$
loss rate = $\varphi = \frac{\text{precip rate}}{\text{cloud water depth}} = \frac{\text{precip rate} \times \rho_l}{\text{cloud depth} \times LWC} = \frac{4.4 \times 10^{-3} \times 1000}{(1000)(0.4 \times 10^{-3})} \simeq 10/\text{hr}$

This leads to a time scale of about 6 min. Well this is interesting, because it indicates that the cloud exhausts itself of water 10 times an hour. How then is it possible for us to see a cloud for longer than only a few minutes?

Another interesting conclusion is that $\varphi$ is the same as the loss rate of CCN from the entire boundary layer. How is this the case?