Here several important variables are introduced and their relevance on thermodynamic charts explained.

1 Dry variables

The dry static energy includes the internal energy from both sensible heat and the geo-potential a parcel has from being in a gravitational field. In the absence of heating, this quantity is constant, so we give it a name, the dry static energy (units joules per kilogram air)

\[ h_d = c_p T + gz \]

If no energy enters the system through heating then

\[ dh_d = c_p dT + gdz = 0 \]

giving the dry adiabatic lapse rate

\[ \frac{dT}{dz} = -\frac{g}{c_p} \]

The dry adiabatic lapse rate is \( \Gamma_d = -g/c_p = 9.8^\circ C km^{-1} \). The dry static energy is very useful quantity, because like \( \theta \) it is conserved during adiabatic processes. It is also easy to see from \( h_d \) how a process such as orographic lifting would decrease the temperature of a parcel.

A related expression is the potential temperature. Noting that \( \frac{dp}{dz} = -\rho g \), the expression for dry static energy becomes

\[ \rho dh_d = \rho c_p dT - dp = 0 \]

\[ \rho c_p dT = dp \]

and dividing by \( p = \rho R_d T \)

\[ \frac{c_p}{R_d} d\ln T = d\ln p \]

which, upon integration from initial state \( i \) to final state \( f \) gives us

\[ T_f = T_i \left( \frac{p_f}{p_i} \right)^{R/c_p} \]

We can use the above equation to derive an expression for the potential temperature \( \theta \), which represents the temperature a parcel would have if it were brought adiabatically from from \( (p, T) \) to standard pressure \( p_0 \), which is typically set to 1000 mb. In this case

\[ \theta = T \left( \frac{p_0}{p} \right)^{R/c_p} \]

The potential temperature is an important concept as it represents a line of thermodynamic equilibrium for dry air in which there is no net exchange of energy with the environment through radiative absorption or emission, or through turbulent mixing, such that \( dh_d = 0 \). An air parcel may rise or sink, or move around horizontally, but if there is no energy exchange, \( \theta \) is constant. Effectively, all parcels lying along a constant theta surface are in local thermodynamic equilibrium (Fig. 1).

The potential temperature is very useful for being a tracer of air motions in systems that are not changing rapidly. We will discuss this more later but consider the following time scales for energy exchange.
Figure 1: Zonal mean temperature and potential temperatures in the troposphere (From Houghton, 3rd Ed.).

- Free troposphere with no condensational processes – 1 to 2 weeks is the timescale for radiative cooling to have an influence
- Synoptic disturbances: timescale – 1 day
- Convection: timescales – minutes to hours

The implication is that an assumption of thermodynamic equilibrium, and motion along constant $\theta$ surfaces is a useful guide over shorter time scales in non-convective environments.

## 2 Water Vapor and Thermodynamics

So far we have neglected the role of water vapor in our discussions. Why is water vapor important when it is just a small component of the atmosphere? There are two possible reasons

1. The specific heat $c_{pv}$ of water vapor is about twice ($1952 \text{ J deg}^{-1}\text{kg}^{-1}$), that of dry air, meaning it will take twice as much energy to raise the temperature of pure water vapor by comparison.

2. Water vapor pulls of the convenient trick of condensing with an accompanying release of latent heat.

In the atmosphere really only the second is all that important. Assuming a high mixing ratio of 10 g/kg:

$$\bar{c}_p = (1 - w) c_p + w c_{pv} = 1013 \text{ J/kg}$$

which for most purposes is only $9 \text{ J/kg}$ ($\sim 1\%$) different than $c_p$ for dry air. However if we assume only one tenth of the water vapor condenses, the release of latent heat is $\chi L_v$, where $L_v = 2.5 \times 10^6 \text{ J/kg}$, and $\chi = 0.1w$ is the mixing ratio of condensate. This translates to $2.5 \times 10^3 \text{ J/kg}$. A lot! Obviously then phase changes are a major process controlling the thermodynamics of the atmosphere.
So if we include the potentially available energy from water vapor condensation, we adjust the dry static energy to get the moist static energy

\[ h_m = c_p T + g z + L_v r \]

In a moist adiabatic process

\[ dh_m = c_p dT + g dz + L_v dr_s = 0 \]

We can use this to derive the saturated adiabatic lapse rate, along basically the same lines as our derivation of the dry adiabatic lapse rate

\[ \Gamma_s \equiv -\frac{dT}{dz} = \frac{g}{c_p} + \frac{L_v}{c_p} \frac{dr_s}{dz} \]

but \( r_s \) is a function of temperature and pressure since \( r_s \simeq \epsilon e_s(T) / p \)

\[ \frac{dr_s}{dp} = \frac{\partial r_s}{\partial T} \frac{dT}{dp} \]

By substitution, we get

\[ \Gamma_s = \frac{g}{c_p} \left( 1 - \rho L \left( \frac{\partial r_s}{\partial T} \right)_T \right) \]

And recognizing \( r_s \simeq \epsilon e_s(T) / p \), and the C-C equation for \( e_s(T) \) we get

\[ \Gamma_s \equiv -\frac{dT}{dz} = \frac{g}{c_p} \left( 1 + \frac{L r_s}{c_p} \frac{\partial r_s}{\partial T} \right) \]

In some texts (including Edition 3 of Holton and the old Wallace and Hobbs) it is assumed that \( dr_s = \left( \frac{\partial r_s}{\partial T} \right)_T dT \), ignoring the dependence of \( r_s \) on pressure. This leads to very wrong answers.

A related variable is the equivalent potential temperature \( \theta_e \). The equivalent potential temperature takes into account the latent heat release associated with adiabatic lifting, under an assumption that water vapor that condenses in a rising air parcel, releases latent heat while the condensate instantly precipitates. This last component is important because condensed water has a different specific heat and it has a gravitational drag force. In other words it is assumed all water that condenses immediately falls out. This disappearance of the liquid water makes processes along lines of constant equivalent potential temperature \( \theta_e \), or pseudo-adiabats, irreversible, where

\[ \theta_e = T \left( \frac{p_0}{p} \right)^{R/c_p} \exp \left[ \frac{r_s L_v}{c_p T} \right] \]

\[ \theta_e = \theta \exp \left[ \frac{r_s L_v}{c_p T} \right] \]
Potential temperature by example

An air parcel rises from the surface at 1000 mb to 700 mb without mixing with its surroundings or being cooled or heated radiatively. If it’s surface temperature is 25 C, what is its final temperature?

From the following we get

\[ T_f = T_i \left( \frac{p_f}{p_i} \right)^{R/c_p} = 269 \text{ K} \]

Actually this equation can be related to the dry adiabat on a Skew T log p plot, if we consider from above that

\[ \ln T = R/c_p \ln p + \text{const} \]

On the Skew T log p plot ln p is the ordinate. Since T not ln T is the abscissa, adiabats are not straight but slightly curved from the lower right to upper left of the diagram On a Skew-T the sloping black lines are for constant temperature and horizontal black lines are for constant pressure. The sounding for temperature and dewpoint (the temperature \( T_d \) at which \( e(T) \rightarrow e_s(T_d) \) are thick red and blue, respectively. Dry-adiabats (brown lines) are the same as lines of constant \( \theta \). Pseudo-adiabats, or lines of constant equivalent potential temperature \( \theta_e \) are purple lines. Constant saturation mixing ratio \( r_s \) are dashed green lines.

Note that in this particular plot, the temperature lapse-rate is zero to positive at the surface, and the potential temperature increases by about 5 K, i.e. there is an atmospheric inversion. This is a
very stable situation that will greatly hinder convection driven by surface heating. However, notice
the lapse rate is very nearly dry adiabatic just above the inversion. Only a small perturbation in
temperature would be enough to cause vertical mixing. Indeed, between 800 and 650 mb there is
essentially a dry “mixed-layer”, in which potential temperature is constant with height. Another
clue that this is a mixed-layer is that the dewpoint follows a line of constant $r_s$, suggesting parcels
that are lifted upwards without mixing with a drier environment.

Where $T$ and $T_d$ nearly intersect at 650 mb we see the lifting condensation level (or LCL), and
the temperature and dewpoint fall along pseudo-adiabats, suggesting a solid cloud deck. Above
550 mb temperature continues to fall nearly pseudo-adiabatically up to 400 mb, but the dewpoint
becomes drier, suggesting perhaps the entraining tops of a cumulus cloud capped by another tem-
perature inversion just above.