Parcel Model

Meteorology 5210

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1 Governing Equations for Precipitating Convection

For precipitating convection, we have the following set of equations for potential temperature, θ, mixing ratios of water vapor, w, cloud water, l, and rainwater, r;

$$\frac{d\theta}{dt} = \frac{L}{c_p\bar{\pi}}(C - E_r) + D_\theta$$

$$\frac{dw}{dt} = -(C - E_r) + D_w$$

$$\frac{dl}{dt} = C - A_r + D_l$$

$$\frac{dr}{dt} = \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}(\bar{\rho}Vr) - E_r + A_r + D_r$$

The dynamics are governed by the equations for the $x-$, $y-$, and $z-$ velocity components, U, V, and W; and non-dimensional pressure perturbation, π_1:

$$\frac{dU}{dt} = -c_p\bar{\theta}_v\frac{\partial\pi_1}{\partial x} + D_U$$

$$\frac{dV}{dt} = -c_p\bar{\theta}_v\frac{\partial\pi_1}{\partial y} + D_V$$

$$\frac{dW}{dt} = -c_p\bar{\theta}_v\frac{\partial\pi_1}{\partial z} + g\left(\frac{\theta - \bar{\theta}}{\bar{\theta}} + 0.61(w - \bar{w}) - l - r\right) + D_W$$

$$\frac{\partial\pi_1}{\partial t} = -\frac{c_s^2}{c_p\bar{\theta}_v{}^2}\left[\frac{\partial}{\partial x}(\bar{\theta}_v U)\frac{\partial}{\partial y}(\bar{\theta}_v V) + \frac{\partial}{\partial z}(\bar{\theta}_v W)\right].$$

where θ_v is the virtual potential temperature, C is the net condensation rate, E_r is the rain evaporation rate, A_r is the cloud-to-rain water conversion rate, and D_i represents the effects of turbulent mixing. Overbars indicate hydrostatic, reference state values. The hydrostatic reference state obeys

$$\frac{d\bar{\pi}}{dz} = -\frac{g}{c_p \bar{\theta}_v},$$

where π, the nondimensional pressure (Exner function), is defined as

$$\pi \equiv \left(\frac{p}{p_0}\right)^{R/c_p}$$

where p_0 is a constant reference pressure. T is given by

$$T = \bar{\pi}\theta.$$

2 Parcel Model

The parcel model assumes that convection consists of discrete buoyant parcels, often called thermals. The goal of the parcel model is to predict the *average* properties of such thermals. In the simplest versions of the parcel model, we assume that $\pi' = 0$, and that any rain falls out of the parcel immediately. In the parcel model, the turbulent mixing terms D_i represent entrainment of environmental air.

The resulting equations for the average properties of a parcel are:

$$\frac{d\theta}{dt} = \gamma C + D_\theta \tag{1}$$

$$\frac{dw}{dt} = -C + D_w \tag{2}$$

$$\frac{dl}{dt} = C - A_r + D_l \tag{3}$$

$$\frac{dW}{dt} = g\left(\frac{\theta - \bar{\theta}}{\bar{\theta}} + 0.61(w - \bar{w}) - l\right) + D_W \tag{4}$$

where $\gamma \equiv L/(c_p \bar{\pi})$.

This set governs the properties of an air parcel as it undergoes ascent or descent. The vertical velocity equation determines the rate of ascent or descent of the parcel, and therefore its height as a function of time. We can determine the parcel's pressure from its height using the hydrostatic equation since $\pi' = 0$.

3 Saturation Adjustment

In general, this set of equations must be solved (or integrated) numerically. Given the initial parcel properties, we use the differential equations to calculate the change of the parcel properties over a small time interval. We then use the calculated changes to update the parcel properties. This process is repeated as many times as desired.

Each small time interval, or *time step,* corresponds to a change in the parcel's pressure from p^n to $p^{n+1} = p^n + \Delta p$ where n is the old time level and $n+1$ denotes the new time level. For computational purposes, we consider the changes that occur in the parcel's properties during each time step to take place in two stages:

1. All processes operate except phase changes involving cloud droplets ($C = 0$).

2. Only phase changes involving cloud droplets operate ($C \neq 0$).

This two-stage process is very nearly thermodynamically equivalent to the actual, continuous process.

Let the values of θ, w, and l after the first stage be θ^*, w^*, and l^*. The air may be subsaturated, supersaturated, or exactly saturated at this point. It is observed that supersaturation is very small in real clouds. *We will simply assume that supersaturation does not occur.* We will also assume that cloud droplets evaporate immediately when the relative humidity falls below 100%. In other words, when the air is saturated, $w = w_s(T, p)$, where w_s is the saturation mixing ratio, and when unsaturated, $l = 0$. Saturation adjustment, the second stage, enforces these conditions.

During the saturation adjustment,

$$\Delta\theta = \gamma C \Delta t,$$

$$\Delta w = -C\Delta t,$$

$$\Delta l = C\Delta t,$$

where $\Delta\phi \equiv \phi^{n+1} - \phi^*$, for any variable ϕ, and Δt is the time step. See Fig. 1. The unknown C can be eliminated from this set of equations by forming the following set:

$$\Delta\theta + \gamma\Delta w = 0,$$

$$\Delta w + \Delta l = 0.$$

These imply that:

$$\theta^{n+1} + \gamma w^{n+1} = \theta^* + \gamma w^* \tag{5}$$

$$w^{n+1} + l^{n+1} = w^* + l^*. \tag{6}$$

These two equations express conservation of energy (first law of thermodynamics) and conservation of suspended water mixing ratio (vapor and cloud droplets), respectively. They also form a set of two equations in three unknowns: θ^{n+1}, w^{n+1}, and l^{n+1}. This means that before we can solve for the unknowns, we need to provide another equation.

We stated above that we will assume that when the air is saturated, $w = w_s$, and when unsaturated, $l = 0$. At any given time, only one of these conditions can hold. We first assume that the air will be exactly saturated after adjustment, so that

$$w^{n+1} = w_s(T^{n+1}, p^{n+1}), \tag{7}$$

where $w_s(T, p)$ is the saturation mixing ratio,

$$w_s(T, p) = 0.622 \frac{e_s(T)}{p - e_s(T)}, \tag{8}$$

and $e_s(T)$ is the saturation vapor pressure. One may use Bolton's (1980) formula for $e_s(T)$:

$$e_s(T) = 6.112 \exp\left(\frac{17.67 T_c}{T_c + 243.5}\right), \tag{9}$$

where e_s is in mb, $T_c = T - T_0$, and $T_0 = 273.15$ K.

Equation (7) closes the set (5), (6), and (7). However, this set must be solved iteratively because w_s is a non-linear function of T. To obtain a direct (non-iterative) solution, expand w_s in a Taylor series in T about $w_s(T^*, p^{n+1})$ and neglect all terms of second and higher order:

$$w^{n+1} \approx w_s(T^*, p^{n+1}) + \left(\frac{\partial w_s}{\partial T}\right)_{T=T^*, p=p^{n+1}} (T^{n+1} - T^*). \tag{10}$$

The set (5), (6), and (10) can now be solved algebraically for θ^{n+1}, w^{n+1}, and l^{n+1}.

To solve the set, we first write (10) in terms of θ instead of T:

$$w^{n+1} = w_s^* + \alpha^*(\theta^{n+1} - \theta^*), \tag{11}$$

where $w_s^* \equiv w_s(T^*, p^{n+1})$, $\alpha^* \equiv \alpha(T^*, p^{n+1})$, and

$$\alpha(T, p) \equiv 0.622 \frac{\pi p}{(p - e_s(T))^2} \left(\frac{de_s}{dT}\right)_T. \tag{12}$$

For de_s/dT, one may use the Clausius-Clapeyron equation:

$$\frac{de_s}{dT} = \frac{L e_s}{R_v T^2}, \tag{13}$$

where $L = 2.5 \times 10^6$ J/kg and $R_v = 461.5$ J/(kg K).

Now use (11) in (5) to eliminate w^{n+1}. Then solve for θ^{n+1}:

$$\theta^{n+1} = \theta^* + \frac{\gamma}{1 + \gamma\alpha^*}(w^* - w_s^*). \tag{14}$$

Once θ^{n+1} is known from (14), we can immediately obtain w^{n+1} from (11), and l^{n+1} from (6).

If $w^{n+1} \le w^* + l^*$, then (6) implies that $l^{n+1} \ge 0$. This means that our assumption that the air is saturated is correct. If $w^{n+1} > w^* + l^*$, (6) implies that $l^{n+1} < 0$, which is impossible and means that our assumption of saturation is incorrect. Therefore, the air is not saturated, so

$$l^{n+1} = 0 \tag{15}$$

replaces (10). Then (5) and (6) become

$$w^{n+1} = w^* + l^*, \tag{16}$$

$$\theta^{n+1} = \theta^* - \gamma(w^{n+1} - w^*). \tag{17}$$

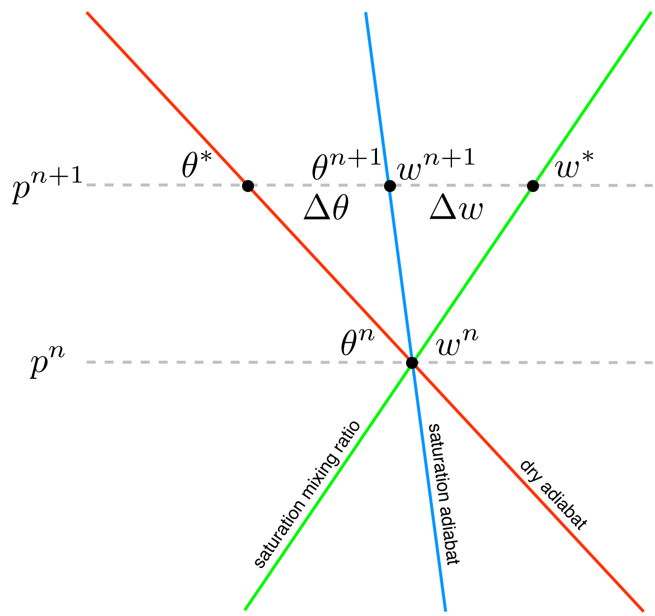

Figure 1: Saturation adjustment.

4 Diabatic Processes

Both adiabatic and diabatic processes affect the thermodynamic properties of a parcel. These properties are governed by (1)-(3). Condensation of water vapor to form cloud droplets and evaporation of cloud droplets to form water vapor are (saturated) adiabatic processes. The net condensation rate is positive when more water is condensing than is evaporating, and negative when more water is evaporating than is condensing. In (1)-(3), the net condensation rate is denoted by C. It is implicitly determined by the saturation adjustment algorithm. The remaining processes, conversion of cloud water to rain (A_r) and entrainment (D_θ, D_w, D_l), are diabatic. They are usually represented explicitly.

In (1)-(3), the process rates are per unit time interval. For example,

$$A_r \equiv \left(\frac{dl}{dt}\right)_{\text{conversion to rain}} = \left(\frac{dr}{dt}\right)_{\text{conversion from cloud water}}.$$

We are often more interested in how the thermodynamic properties of a parcel change with pressure than with how they change with time. The equations that govern the rates of change with pressure of a parcel's thermodynamic properties are obtained from (1)-(3) by dividing by $-dp/dt$:

$$-\frac{d\theta}{dp} = \gamma\hat{C} + \hat{D}_\theta \tag{18}$$

$$-\frac{dw}{dp} = -\hat{C} + \hat{D}_w \tag{19}$$

$$-\frac{dl}{dp} = \hat{C} - \hat{A}_r + \hat{D}_l \tag{20}$$

In (18)-(20), the process rates are per unit *decrease in pressure.*

As before, the net condensation rate, $\hat{C}$, in (18)-(20), is implicitly determined by the saturation adjustment algorithm. The diabatic processes, conversion of cloud water to rain ($\hat{A}_r$) and turbulent mixing ($\hat{D}_\theta, \hat{D}_w, \hat{D}_l$), remain to be specified.

A very simple formulation of the conversion rate of cloud water to rain is

$$-\hat{A}_r \equiv \left(-\frac{dl}{dp}\right)_{\text{conversion to rain}} = -Cl,$$

for $dp/dt < 0$ only, with $C = 2 \times 10^{-2}\ \text{mb}^{-1}$.

4.1 Entrainment

The other terms in (18)-(20) represent the effects of entrainment. Entrainment is the incorporation of environmental air into the parcel (Fig. 2). The *fractional rate of entrainment* of a parcel of mass m that entrains a blob of mass dm while the pressure changes by $-dp$ (due to ascent) is

$$\lambda \equiv -\frac{1}{m}\frac{dm}{dp}.$$

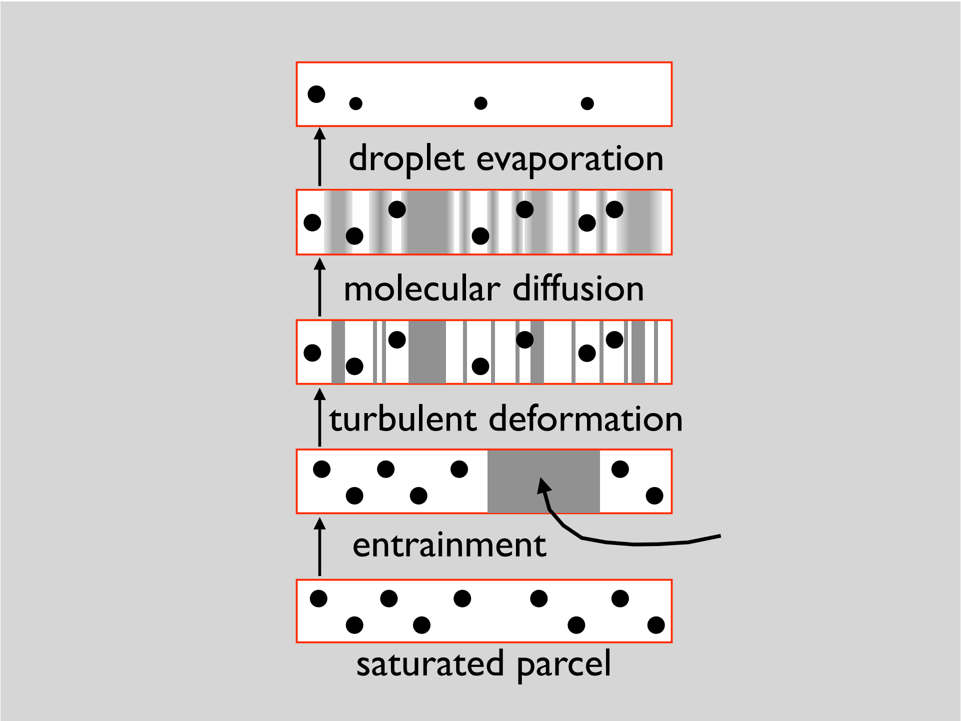

Figure 2: Entrainment.

The rate of change of a scalar ϕ due to entrainment is

$$\hat{D}_\phi \equiv \left(-\frac{d\phi}{dp}\right)_{\text{entrainment}} = -\lambda(\phi - \phi_e), \tag{21}$$

where ϕ_e is the value of ϕ in the entrained air. We can derive (21) from

$$\left(-\frac{d\phi}{dp}\right)_{\text{entrainment}} = \lim_{\Delta p \to 0} \frac{\phi_{\text{after ent}} - \phi_{\text{before ent}}}{-\Delta p} \tag{22}$$

using

$$\phi_{\text{before ent}} = \phi \tag{23}$$

and

$$\phi_{\text{after ent}} = \frac{m\phi + \Delta m\ \phi_e}{m + \Delta m}. \tag{24}$$

Substitution of (23) and (24) into (22) gives

$$\begin{aligned}
\left(-\frac{d\phi}{dp}\right)_{\text{entrainment}} &= \lim_{\Delta p \to 0} \frac{1}{m + \Delta m} \frac{\Delta m}{\Delta p}(\phi - \phi_e) \\
&= \frac{1}{m}\frac{dm}{dp}(\phi - \phi_e) \\
&= -\lambda(\phi - \phi_e).
\end{aligned}$$

By applying (21) to θ, w, and l, we obtain

$$\hat{D}_\theta = -\lambda(\theta - \theta_e), \tag{25}$$

$$\hat{D}_w = -\lambda(w - w_e), \tag{26}$$

$$\hat{D}_l = -\lambda(l - l_e) = -\lambda l. \tag{27}$$

In cumulus clouds, the fractional rate of entrainment, λ, ranges from about 0.1 km^{-1} to 2 km^{-1}. Cloud-top height is largely determined by λ: deep clouds are associated with small values, and shallow clouds with large values. It has been found from field studies that $\lambda \sim 0.2/R$, where R is the cloud radius.