Turbulence Kinetic Energy (TKE) Equation

To study turbulence production we form an eq. for TKE:

(1) Subtract mean momentum eqs. (for $\bar{u}, \bar{v}, \bar{w}$) from eqs. for u, v, w to get eqs. for u', v', w'.

(2) multiply eqs. for u', v', w' by u', v', w', respectively and sum.

(3) Average the result to get an eq. for $\overline{u'^2} + \overline{v'^2} + \overline{w'^2}$, which is $2 \times$ TKE per unit mass.

The resulting eq. is complicated. It can be written symbolically as:

$$\frac{\overline{D}\,(TKE)}{Dt} = \underbrace{MP}_{\substack{\text{mechanical}\\ \text{(shear)}\\ \text{production}}} + \underbrace{BPL}_{\substack{\text{buoyant}\\ \text{production}\\ \text{or loss}}} + \underbrace{TR}_{\substack{\text{redist.}\\ \text{by turb.}\\ \text{transport}\\ \text{\& pressure}\\ \text{forces}}} - \underbrace{\varepsilon}_{\substack{\text{frictional (viscous)}\\ \text{dissipation}\\ >0}} . \qquad (5.14)$$

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<u>BPL</u> : conversion between mean flow potential energy and turbulent K.E.

$$\boxed{\begin{matrix}\text{mean}\\ \text{P.E.}\end{matrix}} \xleftrightarrow{\;BPL\;} \boxed{TKE}$$

For dry air, $BPL = \dfrac{g}{\theta_0} \overline{w'\theta'}$.

Recall that for a single parcel,

$$\frac{1}{2}\left(w_2^2 - w_1^2\right) = \int_{z_1}^{z_2} \frac{g}{\theta_0}\,\theta'\,dz\,,$$

where $\theta' = \theta - \bar{\theta}$.

Divide by $\Delta t = \Delta z / w$; where $\Delta z = z_2 - z_1$; and $w \approx \frac{1}{2}(w_1 + w_2)$.

$$\frac{1}{2}\,\frac{\left(w_2^2 - w_1^2\right)}{\Delta t} = \frac{w}{\Delta z}\int_{z_1}^{z_1+\Delta z} \frac{g}{\theta_0}\,\theta'\,dz\,.$$

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By fundamental theorem of calculus, as $\Delta t, \Delta z \to 0$,

$$\frac{d}{dt}\left(\frac{w^2}{2}\right) = \frac{g}{\theta_0} w \theta'.$$

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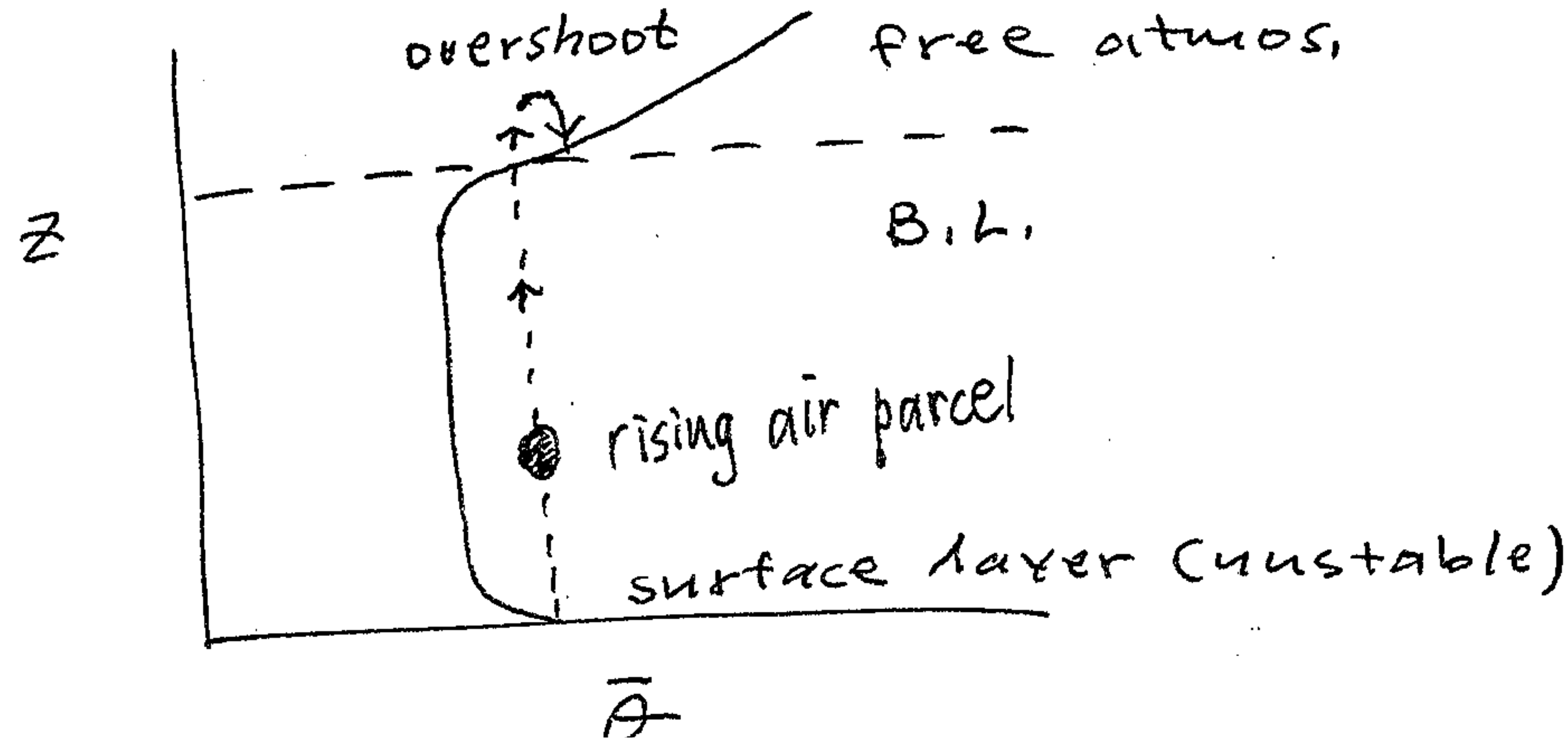
But $w = \bar{w} + w' \approx w'$ since $\bar{w} \approx 0$, so

$$\boxed{\frac{d}{dt}\left(\frac{w'^2}{2}\right) = \frac{g}{\theta_0}\, w'\theta'.}$$

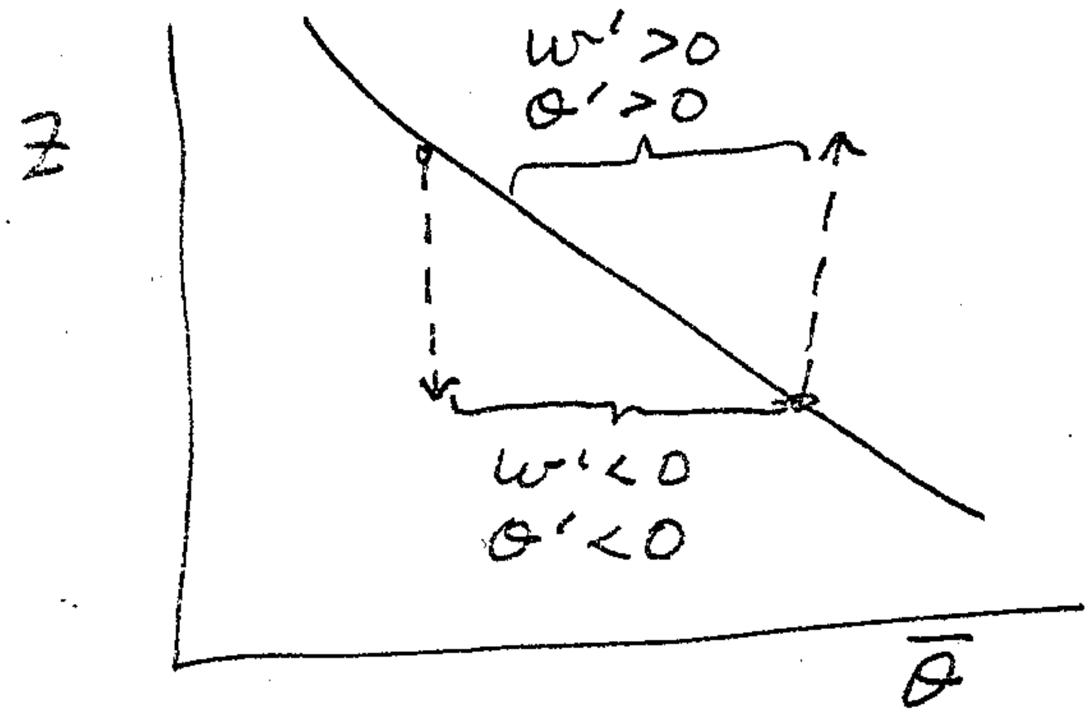
Average to get $\frac{d}{dt}\left(\frac{\overline{w'^2}}{2}\right) = \frac{g}{\theta_0}\,\overline{w'\theta'}$.

Positive Buoyancy prod. occurs when there is heating at surface so an unstable lapse rate develops:

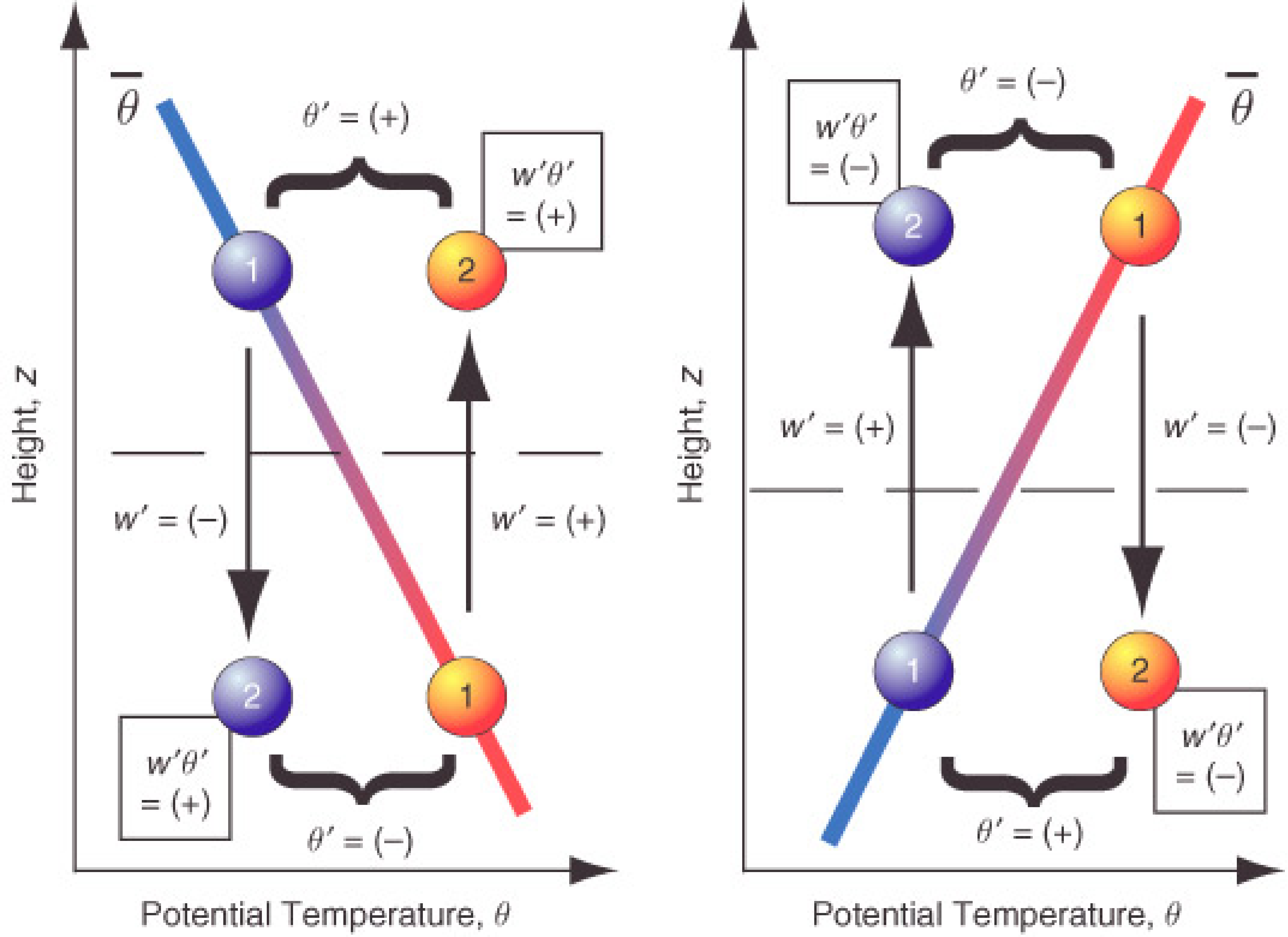
Stull, Fig. 11.11



Notice that in surface layer, $\overline{w'\theta'} > 0$:



If $\overline{\theta}$ profile is stable, $\overline{w'\theta'} < 0$, which reduces or stops turbulence.



(a) Statically unstable: $\partial\overline{\theta}/\partial z < 0$.

(b) Statically stable: $\partial\overline{\theta}/\partial z > 0$.

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<u>MP</u> :

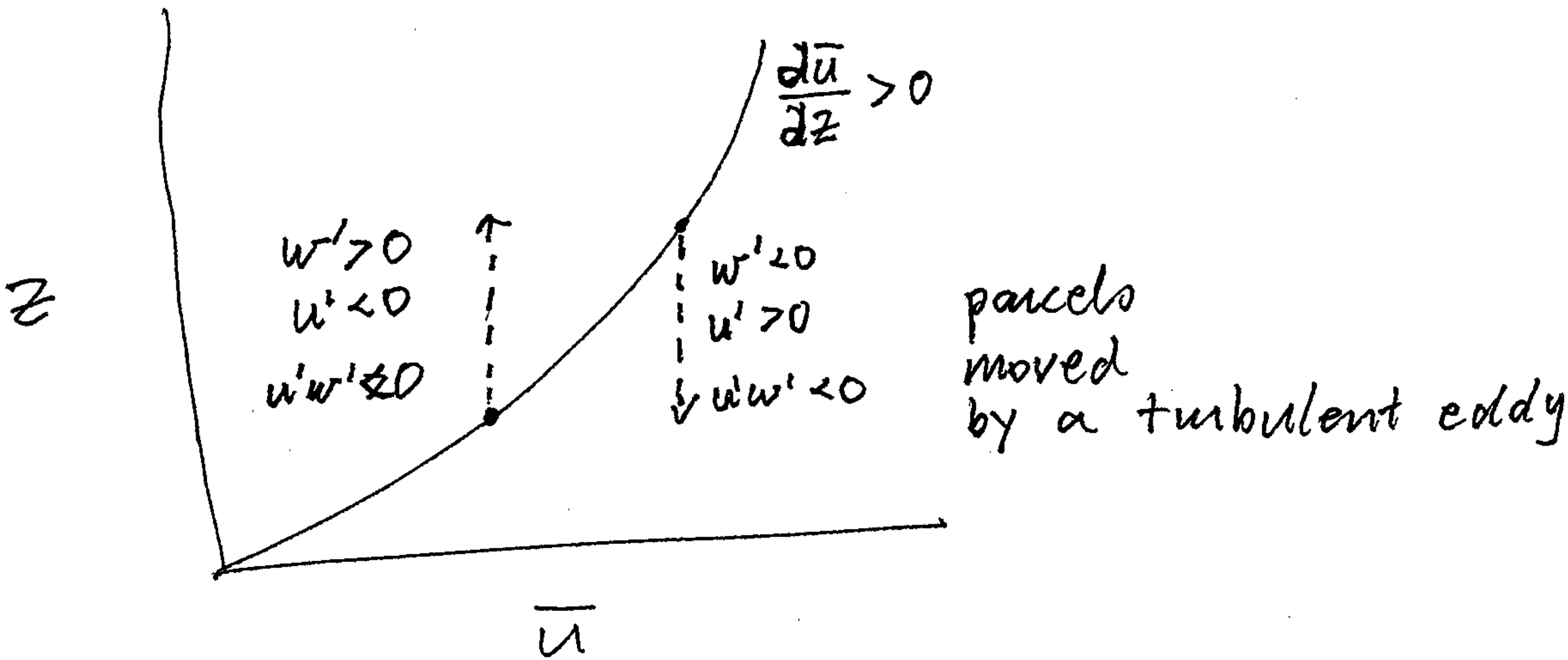
$$\boxed{\begin{matrix}\text{mean}\\ \text{K.E.}\end{matrix}} \xrightarrow[\;MP\;]{} \boxed{\begin{matrix}\text{turbulent}\\ \text{K.E.}\end{matrix}}$$

$$MP = -\overline{u'w'}\,\underbrace{\frac{\partial \bar{u}}{\partial z}} - \overline{v'w'}\,\underbrace{\frac{\partial \bar{v}}{\partial z}}$$

shear

(MP > 0 when momentum flux is down gradient of mean momentum.)

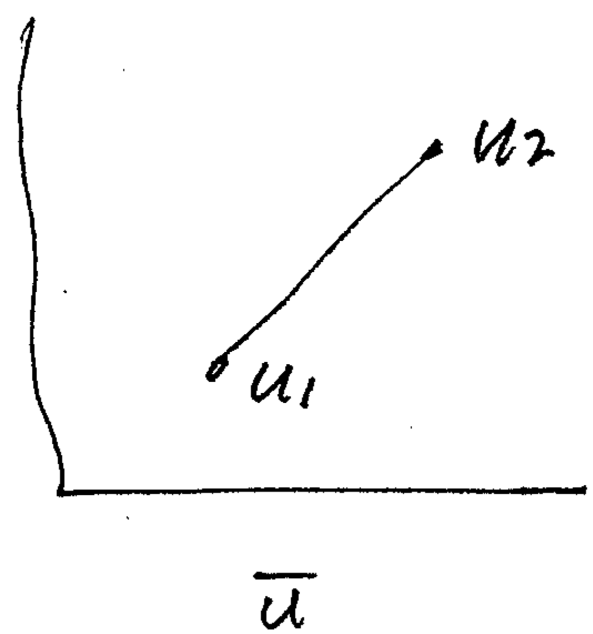
Mechanical (Shear) Production (MP)



momentum is transferred from region of large $\bar{u}$ to region of low $\bar{u}$, i.e., down gradient, while $\overline{u'w'} < 0$ and MP > 0.

Effect of MP on mean wind profile and mean KE

before turbulence



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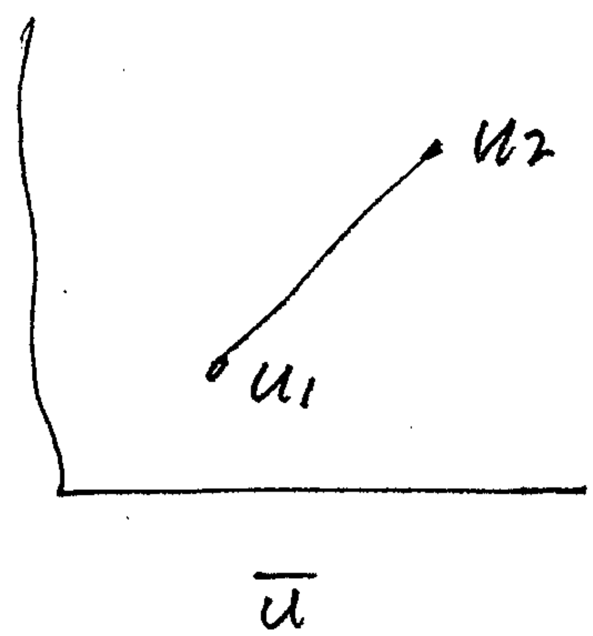
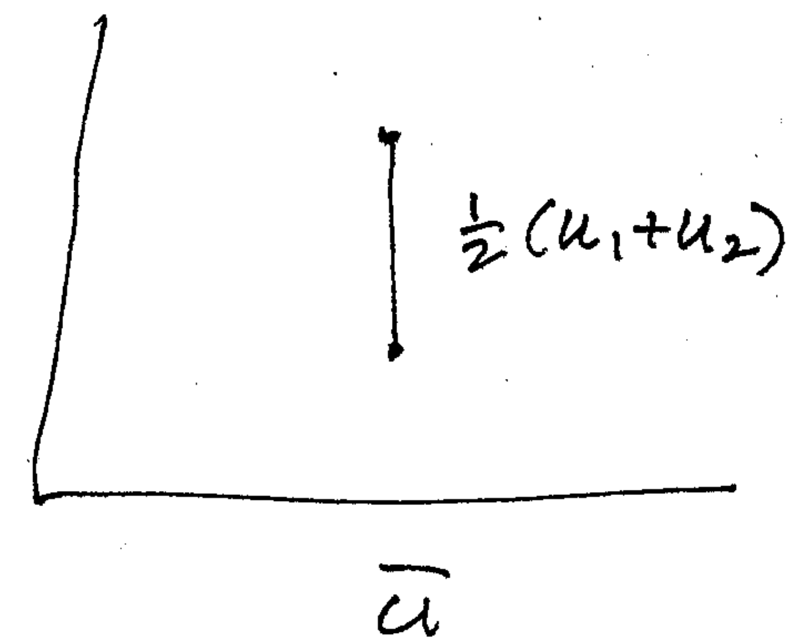


after turbulence



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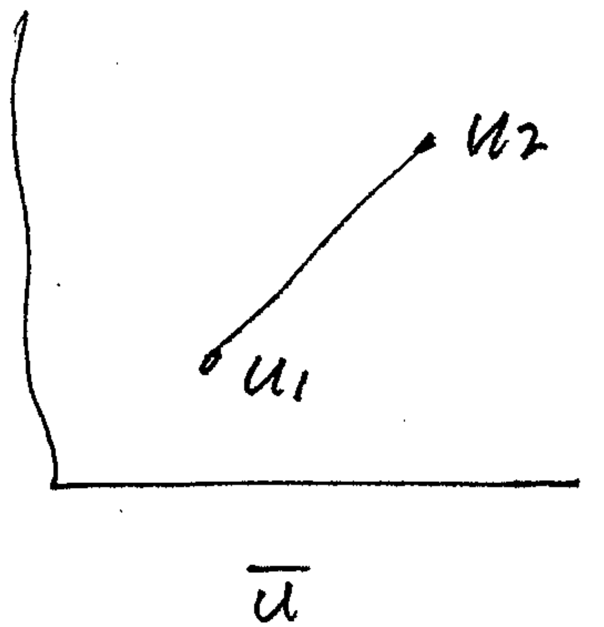
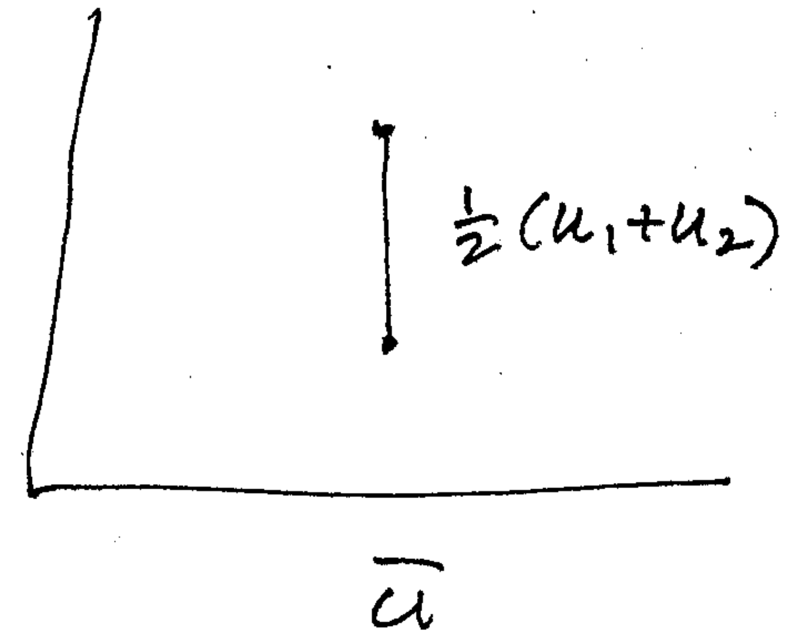


after turbulence



mean K.E. for layer =

$$\boxed{\frac{1}{2}\left(u_1^2 + u_2^2\right)}$$

BEFORE

Effect of MP on mean wind profile and mean KE

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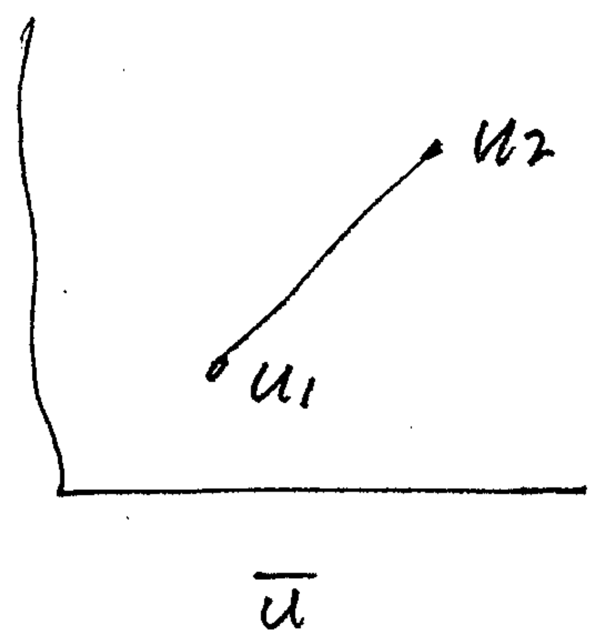
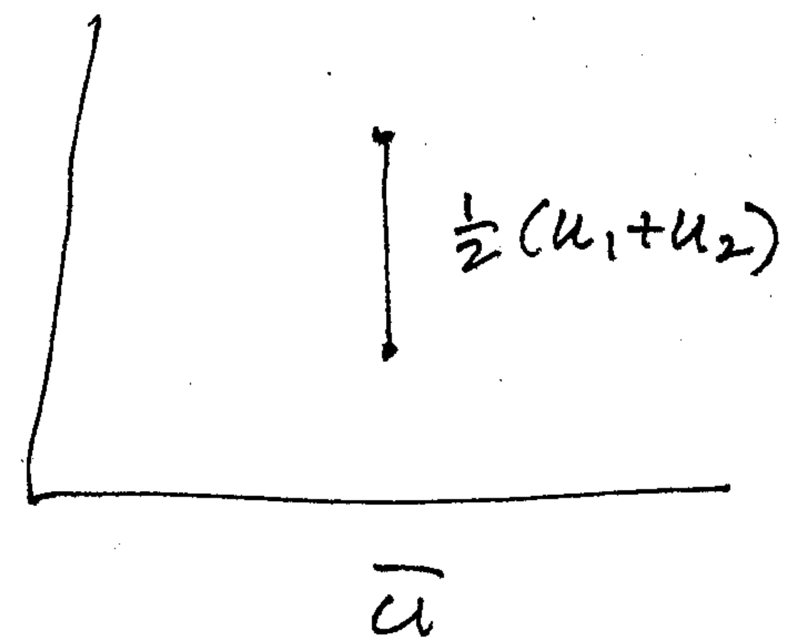


after turbulence



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$$\boxed{\begin{aligned} &2 \times \frac{1}{2}\left[\frac{1}{2}(u_1+u_2)\right]^2 \\ &= \frac{1}{4}\left[u_1^2 + u_2^2 + 2u_1 u_2\right] \end{aligned}}$$

BEFORE **AFTER**

Effect of MP on mean wind profile and mean KE

What is change in mean K.E. for layer?

Before − After =

$$\frac{1}{2}(u_1^2+u_2^2)-\left[\frac{1}{2}(u_1+u_2)\right]^2 = \frac{u_1^2}{2}+\frac{u_2^2}{2}-\frac{u_1^2}{4}-\frac{u_2^2}{4}-\frac{u_1u_2}{2}$$

$$= \frac{u_1^2}{4}+\frac{u_2^2}{4}-\frac{u_1u_2}{2}$$

$$= \frac{1}{4}(u_1^2+u_2^2-2u_1u_2)$$

$$= \frac{1}{4}(u_1-u_2)^2.$$

$\boxed{\text{Thus, regardless of sign of } u_1-u_2\text{, mean K.E. decreases due to mixing by turbulence.}}$

Static Stability, TKE, and Richardson Number

If layer is statically stable, can turbulence exist? only if MP is large enough:
(as measured by <u>flux</u> <u>Richardson</u> <u>number</u>)

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— As static stability increases, depth of turbulent layer decreases.

Static Stability, TKE, and Richardson Number

At night, a strong temperature inversion may be produced by radiative cooling of the surface, and the BL depth may be only a few dm (decameters) deep, since turbulence is suppressed at higher levels where MP is small and BPL ≤ 0.