Atmospheric Sciences 6150
Exercise 1: Buoyancy

1. Emanuel problem 1.1

2. Emanuel problem 1.2

3. Emanuel problem 1.3

   Note that the *stagnation pressure,* p_s, is related to the ambient (i.e., hydrostatic) pressure, $\bar{p}$, at the same level, and to the parcel-relative flow speed far ahead of the parcel, u_0, by Bernoulli's equation (the integral of the momentum equation along a streamline):

$$\frac{p_s - \bar{p}}{\bar{\rho}} = \frac{u_0^2}{2}.$$

4. Emanuel problem 1.4

5. (a) Show that

$$-\frac{1}{\rho}\nabla p = -c_p \theta \nabla \pi,$$

   where π, the *Exner function*, is defined as

$$\pi \equiv \left(\frac{p}{p_r}\right)^{R_d/c_p}.$$

   (b) Derive the hydrostatic equation in terms of $\pi_0(z)$ and $\theta_0(z)$, which are in hydrostatic balance.
   (c) Decompose π and θ into a hydrostatic basic state, $\pi_0(z)$ and $\theta_0(z)$, and deviations from this state, $\pi^{'}$ and $\theta^{'}$, and apply scale analysis, to obtain

$$-c_p \theta \nabla \pi + \mathbf{g} \approx -c_p \theta_0 \nabla \pi^{'} - \mathbf{g}\frac{\theta^{'}}{\theta_0},$$

   where $-\mathbf{g}\theta^{'}/\theta_0$ is the buoyancy acceleration. Recall that $\mathbf{g} = -g\mathbf{k}$.

EXERCISES

1.1 Determine the total buoyancy force acting on a sample of air of dimensions 10^6 m^3 with a uniform temperature of 28°C, immersed in air with a uniform temperature of 0°C. Assume that air is an ideal gas with a gas constant R of 287 J kg^{-1} K^{-1}, at a pressure of 1000 millibars (1 millibar = 10^2 kg m^{-1} s^{-2}). The acceleration of gravity may be taken as 9.8 m s^{-2}. Also determine the force per unit mass acting on the sample.

1.2 Suppose that the buoyancy acceleration acting on the sample in Exercise 1.1 is maintained at a fixed value. Determine the velocity of the sample at altitudes of 1, 2, 3, 4, and 5 km, if it starts from rest at $z = 0$ km.

1.3 Estimate (but do not try to calculate exactly) the perturbation pressure gradient acceleration acting on the sample of air described in Exercise 1.1 when it reaches an altitude of 2 km. Assume that the sample has a fixed volume of 10^6 m^3 and that it has a square cross section on horizontal planes passing through the sample. Also assume that the pressure on the upper face of the volume is the *stagnation pressure* (i.e., the pressure the ambient air has if it has no velocity relative to the moving sample), while the pressure on the lower face is the ambient pressure. (Ignore the direct contribution of pressure perturbations to buoyancy.) Do this for the following five cases:

(a) The sample is ten times as tall as it is wide.
(b) The sample is twice as tall as it is wide.
(c) The sample is a cube.
(d) The sample is twice as wide as it is tall.
(e) The sample is ten times as wide as it is tall.

Express your answers as a fraction of the buoyancy acceleration.

1.4 Determine the Mach number of the sample of air described in Exercises 1.1 and 1.2 when it reaches an altitude of 5 km. The *Mach number* is the square of the ratio of the sample's velocity to the speed of sound. Take $\gamma = 1.4$, $T = 0$°C, and $R = 287$ J kg^{-1} K^{-1}. Is the anelastic approximation justified in this case?