Plume with $N^2 = 0$ (p. 28) (top hat profile)

with $N^2 = 0$ in (2.7.7), we get $R^2 w B$ = constant = F/π, F = boundary buoyancy flux.

Solve other eqs. using sols.

$$w = A z^n \qquad R^2 w B = F/\pi$$
$$R = C z^l$$

Determine A, C, n, l.

Subst. into (2.7.5), (2.7.6):

(2.7.5) $\rightarrow$ $\dfrac{d}{dz}\left(AC^2 z^{2l+n}\right) = 2CA\alpha z^{n+l}$

(2.7.6) $\rightarrow$ $\dfrac{d}{dz}\left(A^2C^2 z^{2l+2n}\right) = \dfrac{F}{\pi A z^n}$

use $B = F/(\pi R^2 w)$, so $R^2 B = \dfrac{R^2 F}{\pi R^2 w} = \dfrac{F}{\pi w}$

Carry out differentiation:

$$AC^2(2l+n)\, z^{2l+n-1} = 2CA\alpha\, z^{n+l}$$

$$A^2C^2(2l+2n)\, z^{2l+2n-1} = \frac{F}{\pi A z^n}$$

Require exponents and coefficients to match:

$$\left.\begin{aligned} AC^2(2l+n) &= 2CA\alpha \\ 2A^2C^2(l+n) &= F/\pi A \\ 2l+n-1 &= n+l \\ 2l+2n-1 &= -n \end{aligned}\right\} \boxed{\text{4 eqs., 4 unknowns}}$$

Solve last two first:

$$\cancel{2}l + \cancel{n} - 1 = \cancel{n} + \cancel{l} \Rightarrow \boxed{l = 1}$$
$$2\cancel{l} + 2n - \cancel{1} = -n$$
$$1 + 3n = 0 \rightarrow \boxed{n = -1/3}$$

2

Use $l = 1$, $n = -1/3$:

$$\cancel{A} C^{\cancel{2}} (2 - 1/3) = 2 \cancel{C} \cancel{A} \alpha \Rightarrow \boxed{C = \frac{2\alpha}{5/3} = \frac{6}{5}\alpha}$$

$$2A^2 \left(\frac{6}{5}\alpha\right)^2 (1 - 1/3) = F/\pi A$$

$$A^3 = \frac{F}{\pi} \frac{1}{2(2/3)(6/5)^2 \alpha^2 \cancel{(6/5)(5/6)\alpha/\alpha}}$$

$$\cancel{2 \cdot \frac{2}{3} \cdot \frac{6}{5} \cdot \frac{6}{5} = \frac{48}{25}}$$

$$= \frac{F}{\pi} \frac{(5/6\alpha)^3}{2 \cdot \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{1}{\alpha}}$$

$$= \frac{F}{\pi} \left(\frac{5}{6\alpha}\right)^3 \left(\alpha \frac{9}{10}\right)$$

so $\boxed{A = \frac{5}{6\alpha} \left(\frac{F}{\pi} \frac{9}{10} \alpha\right)^{1/3}}$

Summary: $l = 1$, $n = -1/3$, $A = \frac{5}{6\alpha}\left(\frac{9}{10}\alpha \frac{F}{\pi}\right)^{1/3}$, $C = \frac{6}{5}\alpha$.

In particular, $R = C z^l = \frac{6}{5} \alpha z$.

~~(Shortened in lecture)~~

when stably stratified ($N^2 > 0$): (p. 29–30)

$w = 0$ at some z,
$B = 0$ at some lower z.

Simple similarity sols. like those for $N^2 = 0$ do not apply.

must solve the governing, radially (area) integrated eqs.

Morton, Taylor, & Turner (1956): obtained a numerical sol., but used eqs. integrated over a Gaussian radial profiles of w, B,

Resulting eqs. differ only in coefficients:

mass flux : $\frac{d}{dz}(R^2 w) = 2\alpha R w$ (n.c.) (2.7.9)

momentum flux : $\frac{d}{dz}(R^2 w^2) = \underline{\underline{2}} R^2 B$ (2.7.10)

Heat flux : $\frac{d}{dz}(R^2 w B) = -\underline{\underline{2}} R^2 w N^2$ (2.7.11)

Here, w, B are values at $r=0$.
now solve these numerically.
First, change variables:

$V \equiv R w$
$U \equiv R^2 w$
$F = R^2 w B$

(2.7.9) − (2.7.11) become

(2.7.12) $\frac{dU}{dz} = 2\alpha V$

(2.7.13) $\frac{dV^4}{dz} = 4FU$

(2.7.14) $\frac{dF}{dz} = -2UN^2$

$\rightarrow$ $R^2w^2 = V^2$, so (2.7.10) $\rightarrow$ $\frac{d}{dz}V^2 = 2R^2B$, and
$dV^4/dz =$
$$2V^2\frac{dV^2}{dz} = (2V^2)2R^2B = 2R^2w^2\,2R^2B = 4(R^2wB)(R^2w) = 4FU$$

B.C. $R = W = 0$, $V = U = 0$ at $z = 0$; also $F = \frac{2}{\pi}F_0$.
F_0 is boundary flux of buoyancy.
Note that

$$F_0 = \int_0^{2\pi}\int_0^{\infty} \hat{w}(r)\,\hat{B}(r)\,r\,dr\,d\theta \qquad (1)$$

where

$$\hat{w}(r) = w\exp(-r^2/R^2)$$

and

$$\hat{B}(r) = B\exp(-r^2/R^2).$$

Subst. in (1) and let $x = r^2/R^2$ so $r\,dr = \frac{R^2}{2}dx$.
Then (1) becomes

$$\pi R^2 wB\int_0^{\infty} e^{-2x}\,dx = \pi R^2\frac{wB}{2} = \frac{\pi}{2}F.$$

Only 2 external parameters, F_0, N^2.
These must determine character of plume.

Simplify eqs. by non-dimensionalizing
(so as to remove parameter dependence):

$$z^* = 2^{-7/8}\,\pi^{-1/4}\,\alpha^{-1/2}\,F_0^{1/4}\,N^{-3/4}\,z$$

$$V^* = 2^{3/4}\,\pi^{-1/2}\,F_0^{1/2}\,N^{-1/2}\,V$$

$$U^* = 2^{7/8}\,\pi^{-3/4}\,\alpha^{1/2}\,F_0^{3/4}\,N^{-5/4}\,U$$

$$F^* = 2\pi^{-1}\,F_0\,f$$

$*$ denote dimensional values.

Dimensional analysis gives exponents for F_0, N. Other exponents are chosen simply to eliminate all coefficients involving $2, \pi$, or α.

(2.7.12)–(2.7.14) become

$$\frac{dU}{dz} = V \quad , \quad \frac{dV^4}{dz} = fU \quad , \quad \frac{df}{dz} = -U \qquad (2.7.15)$$

b.c. $U = V = 0$, $f = 1$ at $z = 0$.

Solve for U, V, f numerically, then get dimensionless forms of w, R, B.

<u>Ex.</u> $\cancel{\dfrac{V^*}{z^*}} = \cancel{\dfrac{R\,w^*}{z^*}} = \cancel{\dfrac{2^{3/4}\,\pi^{-1/2}\,F_0^{1/2}\,N^{-1/2}\,V}{2^{-7/8}\,\pi^{-1/4}\,\alpha^{-1/2}\,F_0^{1/4}\,N^{-3/4}\,z}}$

Get non-dimensional radius:

Note that $R^* = U^*/V^*$, (dimensional)

Define $R \equiv U/V$. (non-dimensional).

Similarly, $w^* = V^{*2}/U^*$,

so $w \equiv V^2/U$,

And $B^* = F^*/U^*$,

so $B \equiv f/U$.

* Show Fig. 2.7. Shows non-dimensional
R, w, B vs. z.

Note that $w = 0$ at $z = 2.8$,
$B = 0$ at $z = 2.125$. (non-dim.)

Parcels overshoot neutral buoyancy level, then decelerate to zero velocity while spreading away from central axis.

(Skipped)
Turbulent thermal in stratified fluid
Analytic solutions may be obtained.
Spherical, radius R.

Similar procedure:
1. Integrate radially over volume.

Mass: $$\frac{d}{dt}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2 \alpha w$$

Momentum: $$\frac{d}{dt}\left(\frac{4}{3}\pi R^3 w\right) = \frac{4}{3}\pi R^3 B$$

Heat: $$\frac{d}{dt}\left(\frac{4}{3}\pi R^3 B\right) = -\frac{4}{3}\pi R^3 w N^2$$

Neutral case
when $N^2 = 0$, $R^3 B$ is constant.
Can solve other eqs. using

$$R = A t^n$$
$$w = C t^m.$$

Sols. have same form as those derived by dimensional analysis (Sec. 2.4).

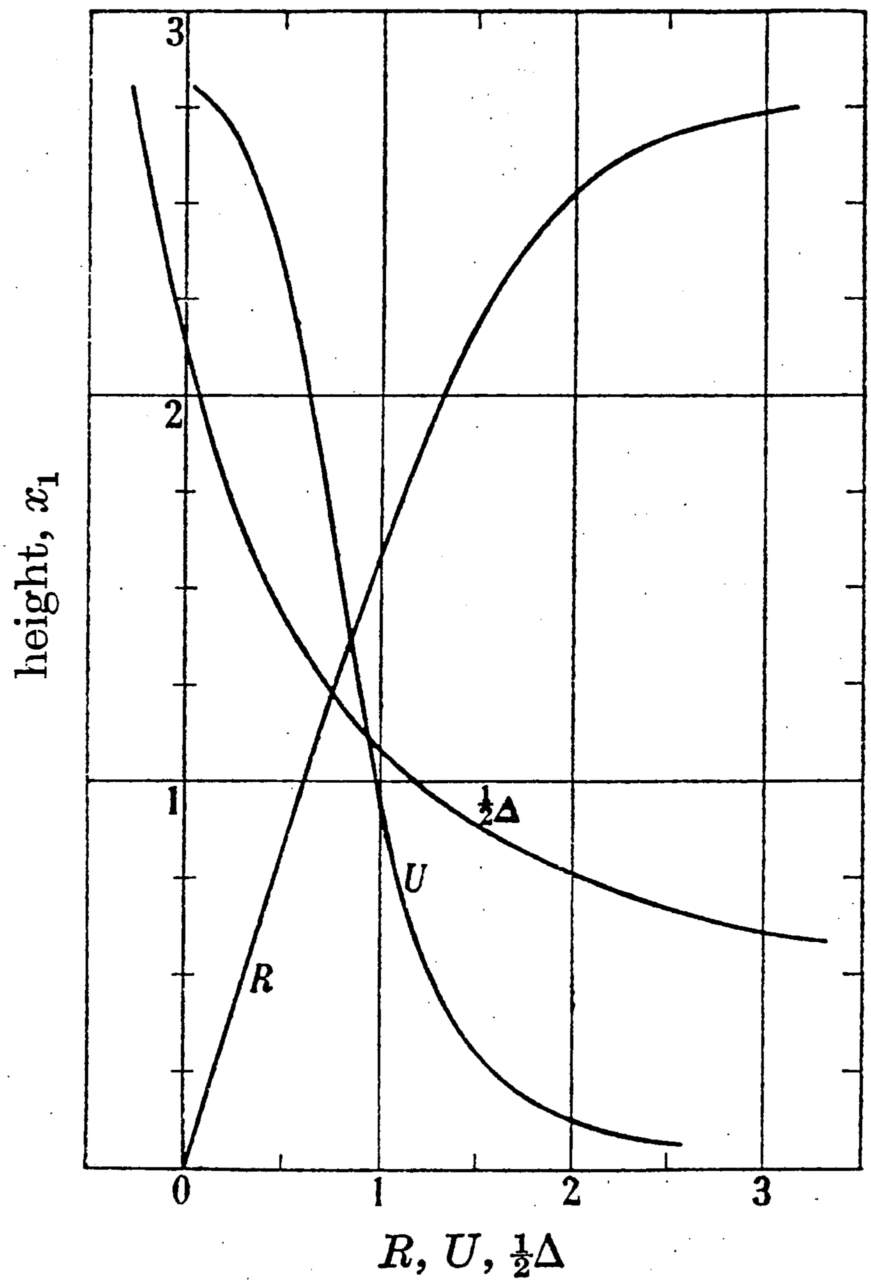

Fig. 2.7 The height dependence of the dimensionless forms of the horizontal extent (R), vertical velocity (U), and buoyancy (Δ) for a turbulent plume in a stably stratified ambient fluid. [*(After Morton, Taylor, and Turner (1956).*]