Stable case ($N^2 = \text{constant} > 0$); can solve set analytically.

2. As before, use new variables:
$$M \equiv R^3 w \, , \quad V \equiv R^3 \, , \quad F \equiv R^3 B .$$

At $t = 0$: $\quad M = V = 0 , \quad F = F_0 \left(= \frac{3Q}{4\pi} \right),$
where
$$Q \equiv \iiint B_0 \, d\tau .$$

3. Render dimensionless (the resulting eqs.)

skip
$$\begin{cases} F^* = F_0 f \\ M^* = \frac{3}{4\pi} F_0 N^{-1} m \\ V^* = \left(\frac{3}{\pi}\right)^{3/4} \alpha^{3/4} F_0^{3/4} N^{-3/2} v \end{cases}$$
$$t^* = N^{-1} t .$$

$*$ dimensional

Obtain dimensionless forms of gov. eqs. (see text, Eqs. (2.7.19) – (2.7.21)).

Can get ht. of thermal from integration of $dz/dt = w$.
Scale z:
$$\frac{M^*}{V^*} t^* = z^* = \frac{1}{4} \left(\frac{3}{\pi}\right)^{1/4} \alpha^{-3/4} F_0^{1/4} N^{-1/2} z$$
$= w^* t^*$

Recall that $w^* = M^*/V^*$, so $w = m/v$ (non-dim.); as expected,
$$\frac{dz}{dt} = \frac{m}{v} \quad \text{(non-dim.)} . \qquad (2.7.22)$$

4. Apply b.c.s to solve

Sols. for R, w, B are obtained, with new b.c.s, sequentially for each segment of upward and downward motion, i.e., for
$0 \le t \le \pi$ (non-dim.):

$z = 4(1-\cos t)^{1/4}$
(see text for others.)

* See Fig. 2.8 Damped oscillatory motion.
max. (dimensionless) ht. of 4.75, eventually asymptotes to 4.2.

Sec 2.B Experiments and Observations

* Show Fig. ~~2~~.9 - plumes in neutral & stable stratif.

* Show Fig. 2.10 - plume in stable stratif.
Shows expected dependence on $F_0^{1/4} N^{-3/4}$ of plume ht.

* Show Fig. 2.11 - Thermal max. ht. in stable stratif. Shows expected dependence on $F_0^{1/4} N^{-1/2}$.

* Show Fig. 2.14 Shape preservation of thermal in neutral envir.

* Fig. 2.15 z^2 vs t shows expected linear relationship deduced from dim. anal.

* Fig. 2.16 circulation within a thermal: "spherical vortex."

H.W. - Do exercises 2.1 - 2.4
For 2.3, use $d = 0.093$.

2.8 Experiments and observations

In analytic solutions, as well as numerical ones, α remained an undetermined constant, to be somehow determined from experiment.

For alternate approach (p. 5)

[Plumes in a stable stratification can be used because the lower part of the plume is essentially the same as that of a similar plume when $N^2 = 0$, (see Figs. 2.7, 2.9).]

From measured heights of a plume in a stable stratification plotted vs. $F_0^{1/4} N_1^{-3/4}$ as in Figs. 2.10, we can determine the constant α, since we saw in § 2.7 that ht. depends only on α, F_0, and N.

Specifically, for a plume,

$$z^* = 0.410\ \alpha^{-1/2} F_0^{1/4} N^{-3/4} z \quad (p. 30)$$

where z is the n.d. ht. of some feature, such as max. plume ht., and z^* is dimensional ht. (measured).

We found that $z = 2.8$ at plume top, so

$$z^*/(F_0^{1/4} N^{-3/4}) = 0.410\ \alpha^{-1/2} (2.8).$$

Actual slope is 3.79, so $\alpha^{-1/2} = \dfrac{3.79}{2.8 \times 0.410}$ or

$$\alpha = \left(\frac{2.8 \times 0.410}{3.79}\right)^2 = 0.092.$$

Similarly, for a thermal,

$$z^* = \frac{1}{4}\left(\frac{3}{\pi}\right)^{1/4} \alpha^{-3/4} F_0^{1/4} N^{-1/2} z \quad (p. 33)$$

ultimate n.d. ht. is $z = 4.2$. For z^* vs. $F_0^{1/4} N^{-1/2}$, Slope $= \frac{1}{4}\left(\frac{3}{\pi}\right)^{1/4} \alpha^{-3/4}(4.2) = 2.66$ from exp. (Fig. 2.11), so

$$\alpha^{-3/4} = 2.66 \Big/ \left(\tfrac{1}{4}\left(\tfrac{3}{\pi}\right)^{1/4} 4.2\right) = 2.5627$$

$$\alpha = (2.5627)^{-4/3} = 0.285.$$

Alternate approach for plume to derive α:

Assume a radial profile of $\exp(-pr^2/z^2)$.

Then slope of ht. of plume top vs. $F_0^{1/4} N^{-3/4}$ depends on p, (see Fig. 2.10). Determine p experimentally as best fit to ~~measured~~ measurements.

Relate p to α as follows:

Equate assumed radial dependence,

$$\exp(-r^2/R^2)$$

to

$$\exp(-pr^2/z^2).$$

Then

$$\frac{R}{z} = p^{-1/2} = \frac{6}{5}\alpha$$

according to solution for plumes with $N^2 = 0$. (See discussion on p. 4 for why we can apply this result to stable stratification.)

From Fig. 2.10, $p = 80$ is best ~~choice~~ match.

Then $\alpha = \frac{5}{6} p^{-1/2} = \frac{5}{6}(80)^{-1/2} = 0.093.$