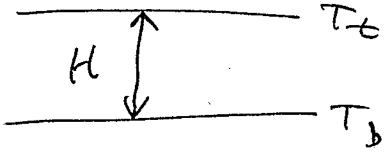
3. Global Convection

Convection in geophysical situations usually originates from widely distributed buoyancy sources.

We now consider behavior of a fluid when the heat source is widely distributed, the Entire fluid is involved in convective overturning – global, not local, convection, results.

Classic problem:



* Benard cells : Fig. 3.1 .

A critical unstable temperature gradient must exist before convection begins. motions have a stationary, cellular character.

Before convection begins, heat is transported only by molecular diffusion.

The stability of this system is determined by

$$Ra \equiv \frac{g \alpha \beta}{\nu K} H^4 \quad \text{Rayleigh no.}$$

α : the maintained temp. gradient $= (T_t - T_b)/H$
H : distance between plates
β : coefficient of thermal expansion:

$$\beta \equiv \frac{1}{\alpha}\left(\frac{\partial \alpha}{\partial T}\right)_p \quad , \text{ and}$$

$$B \simeq g\beta T' \quad (\text{buoyancy due to } T')$$

When $Ra > Ra_c$, convection begins.

The Rayleigh no. is a measure of importance of convective vs. molec. heat transport.

Laminar convection:

momentum eq. is
(neglect p.g.f.)

$$\underbrace{B}_{\text{buoyancy}} + \underbrace{\nu D^2 w}_{\text{viscous drag}} \simeq 0$$

balance

Let w_0 be velocity scale, H length scale, then

$$B \sim \frac{w_0 \nu}{H^2} . \quad \text{or} \quad w_0 \sim \frac{BH^2}{\nu}$$

Ratio of convective to molecular heat flux is Nusselt no.:

$$Nu = \frac{w_0 B}{\kappa B/H} = \frac{w_0 H}{\kappa} \simeq \underbrace{\frac{BH^2}{\nu}\frac{H}{\kappa}}_{\text{use scaling for } w_0} = \underbrace{\frac{g\alpha\beta H^4}{\nu\kappa}}_{\text{use } B \sim g\beta T' \sim g\beta \frac{T'}{H} H \sim g\beta\alpha H} = Ra .$$

Turbulent Convection:

Balance is between buoyancy and fluid accelerations. This implies Froude no. ~ 1, since it is a measure of the relative importance of these terms (see Section 1.3, p. 12).

Thus, $C \equiv \frac{w_0^2}{BH} \sim 1$, or

$$w_0^2 \sim CBH .$$

Then

$$Nu = \frac{w_o B}{\kappa B/H} = \frac{w_o H}{\kappa} = \frac{(CBH)^{1/2} H}{\kappa} = \frac{(Cg\beta\alpha H^2)^{1/2} H}{\kappa}$$

or

$$Nu^2 = \frac{Cg\beta\alpha H^4}{\kappa^2} = \frac{g\alpha\beta H^4}{\nu\kappa} \underbrace{\frac{C\nu}{\kappa}}_{\sigma \, = \text{Prandtl no.}} = C\, Ra\, \sigma$$

Rayleigh no. is also equivalent to the Reynolds no.

$Re = \frac{\text{inertial}}{\text{molecular}}$ accelerations.

Laminar:

(skip deriv.)

$$Re = \frac{w_o H}{\nu} \approx \frac{BH^2}{\nu}\frac{H}{\nu} = (g\beta\alpha H)\frac{H^3}{\nu^2}$$

or

$$Re \simeq \frac{g\alpha\beta H^4}{\nu\kappa}\frac{\kappa}{\nu} = \frac{Ra}{\sigma}. \qquad \boxed{Re = \frac{Ra}{\sigma}}$$

Turbulent:

(skip deriv.)

$$Re^2 = \frac{w_o^2 H^2}{\nu^2} = \frac{CBH^3}{\nu^2} = \frac{Cg\beta\alpha H^4}{\nu^2} = \frac{g\alpha\beta H^4}{\nu\kappa}\frac{C\kappa}{\nu}$$

or

$$\boxed{Re^2 = \frac{Ra}{\sigma}C}$$

Re is measure of stability of flow. Flow becomes turbulent when Re becomes large.

Thus Ra is also a measure of stability; this is validated by experiment.

Exps. show that several regimes of laminar convection are determined by Ra, σ.

§ 3.1 The original Rayleigh problem

Apply linear stability analysis.

skip?

Superpose small disturbance on stationary state. Do they grow?

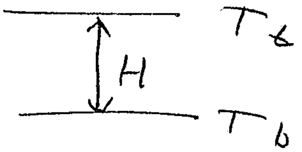
E.g., $u = \bar{u} + \varepsilon u'$ $\varepsilon << 1$.

Solve set of linear eqs. for order ε disturbances.

Goal: Find those configurations of the base state which are unstable, and the structure of the disturbances.

marginal stability: base state parameters for which system first becomes unstable.

Apply this technique to:



Use Boussinesq Navier-Stokes Eqs. (3.1.1)-(3.1.5).

Base state:

$$\bar{u} = \bar{v} = \bar{w} = 0$$
$$\bar{T} = -\alpha z + T_b$$
$$\bar{p} = p_0 - \frac{1}{2} \rho_0 \alpha z^2$$

$p_0 = p(0)$; $\alpha \equiv (T_b - T_t)/H$ = const.

Linearize (3.1.1)-(3.1.5); obtain set

$$\left(\partial/\partial t - \nu D^2 \right) u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \qquad (3.1.6)$$

$$\left(\quad \text{''} \quad \right) v' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \qquad (7)$$

$$\left(\quad \text{''} \quad \right) w' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + B \qquad (8)$$

$$\left(\frac{\partial}{\partial t} - \kappa D^2 \right) B' = \alpha w' \qquad (9)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \qquad (10)$$

$B' \equiv g \beta T'$ (not as in text!).

and $\alpha \equiv g\beta\,(T_b - T_t)/H$ (not in text!)

Now drop primes.

Combine (6)-(10) into a single p.d.e. for w: (see text for details)

$$\left(\frac{\partial}{\partial t} - \kappa D^2\right)\left(\frac{\partial}{\partial t} - \nu D^2\right) D^2 w = \alpha \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w \qquad (13)$$

Non-dimensionalize independent variables:

$$(x^*, y^*, z^*) = H\,(x, y, z)$$

$$t^* = \frac{H^2}{\nu} t$$

$*$: dimensional.

(13) →

$$\left(\sigma \frac{\partial}{\partial t} - D^2\right)\left(\frac{\partial}{\partial t} - D^2\right) D^2 w = Ra \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w \qquad (14)$$

$$\sigma \equiv \frac{\nu}{\kappa}, \qquad Ra \equiv \frac{\alpha H^4}{\nu \kappa}.$$

Assume solution is sum of modes of form:

$$w(x,y,z,t) = w_1(z)\ \exp\{i(k_x x + k_y y) + \omega t\}$$

Then $\frac{\partial}{\partial t} \to \omega$, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \to -k^2 = -(k_x^2 + k_y^2)$

$\nabla^2 \to \frac{d^2}{dz^2} - k^2$, and $\omega = \omega_r + i\omega_i$.

Result is (17):

$$\left[\sigma(\omega_r + i\omega_i) + k^2 - \frac{d^2}{dz^2}\right]\left(\omega_r + i\omega_i + k^2 - \frac{d^2}{dz^2}\right)$$

$$\times\left(k^2 - \frac{d^2}{dz^2}\right) w_1 = Ra\ k^2 w_1 \qquad (17)$$

This o.d.e. (6th order) has 6 b.c.s.
Sols may be found only for special combinations of $\omega_r, \omega_i, k, \sigma, Ra$.

When ω, k are specified, then σ, Ra are <u>characteristic values</u>, of (17).

Find Ra for mode k with $\omega_r = 0$.

Smallest such Ra is critical Ra = Ra_c, first unstable mode as Ra is increased.

Two cases:

$\omega_i = 0$: <u>stationary overturning</u> as $\omega_r > 0$.

$\omega_i \neq 0$ when $\omega_r = 0$: <u>oscillatory instability</u>.

B.C.S   Rigid so $w=0$ at boundaries, since temp. fixed, $B=0$.

Stress-free (free-slip) $\underbrace{\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}}_{\sim \text{stress}} = 0$ at $z=0,1$

From cont. eq., $\partial^2 w/\partial z^2 = 0$ at $z=0,1$,

skip { $\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$ → leads to result above.

Also find that $\partial^4 w/\partial z^4 = 0$ at $z=0,1$,

skip { No slip $u=v=0$ at $z=0,1$.
Cont. eq. → $\partial w/\partial z = 0$
(See text for remainder.)

Sol. for free-slip:

$$w_1(z) = \sum_{n=1}^{\infty} A_n \sin(n\pi z)$$

This satisfies b.c.s.

Subst. into (17): can show that oscillatory instability cannot occur, so $w_i=0$. Also setting $w_r=0$ (marginal stability):

$$\frac{d^2}{dz^2} \to -n^2\pi^2$$

skip { $[k^2 + n^2\pi^2](k^2 + n^2\pi^2)(k^2 + n^2\pi^2) = Ra\, k^2$

or $$Ra = \frac{(k^2 + n^2\pi^2)^3}{k^2}. \qquad (25)$$

For a mode (n, k), this expression gives Ra for which it becomes unstable. Lowest value occurs for $n = 1$:

$$Ra = \frac{(k^2+\pi^2)^3}{k^2} .$$

Find lowest Ra by minimizing w.r.t. k:

$$\frac{\partial Ra}{\partial k} = 0 \rightarrow \boxed{k_c^2 = \frac{\pi^2}{2}} \qquad = \pi/\sqrt{2} \quad (k_c = 2.2214)$$

skip:

$$\frac{\partial Ra}{\partial k} = \frac{3(k^2+\pi^2)^2\, 2k}{k^2} - \frac{2(k^2+\pi^2)^3}{k^3} = 0$$

$$0 = 3\cancel{(k^2+\pi^2)^2}\,\cancel{2}k^2 - \cancel{2}(k^2+\pi^2)^{\cancel{3}} = 0$$

$$0 = 3k^2 - (k^2+\pi^2) = 2k^2 - \pi^2$$

$$\text{or} \quad k^2 = \frac{\pi^2}{2} .$$

skip

$$Ra_c = \frac{(\pi^2/2 + \pi^2)^3}{(\pi^2/2)} = \frac{(\frac{3}{2}\pi^2)^3}{\pi^2/2} = \frac{2\,\frac{27}{8}\pi^6}{\pi^2} = \frac{27}{4}\pi^4$$

$$\boxed{Ra_c = \frac{27}{4}\pi^4 \cong 657.5} \qquad \boxed{\lambda = \frac{2\pi}{k}H = 2\sqrt{2}H = 2.83H}$$

Have found a combined k_c, but don't know how this is partitioned into k_x, k_y. will return to this later.

For both no-slip b.c.s.

$k_c = 3.12$, $Ra_c = 1708$, $\lambda = 2.02\,H$

one free-slip, one no-slip:

$k_c = 2.68$, $Ra_c = 1101$, $\lambda = 2.34\,H$

(See Table 3.1)

Horizontal planforms

$k_c^2 = k_x^2 + k_y^2$. (Any pair that satisfies this be superposed to form any number of geometrical patterns.)

__Expect:__ periodic cells covering domain fully.

This requires cells of equilateral triangles, squares, or hexagons. △ □ ⬡

Rolls (k_x or $k_y = 0$) are also possible.

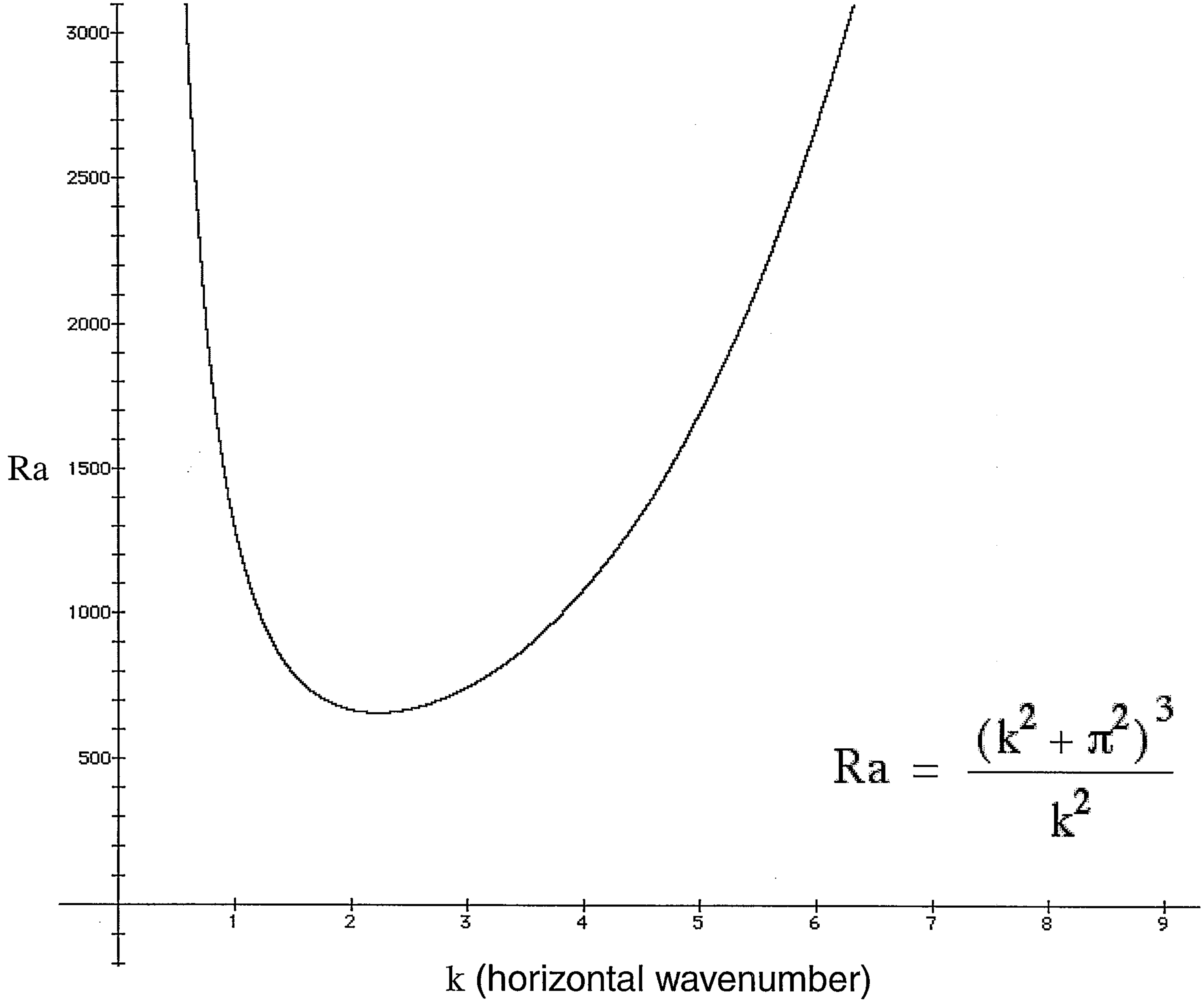
__Fig. 3.2:__ Rectangular cells

__Fig. 3.3:__ Hexagonal cells.

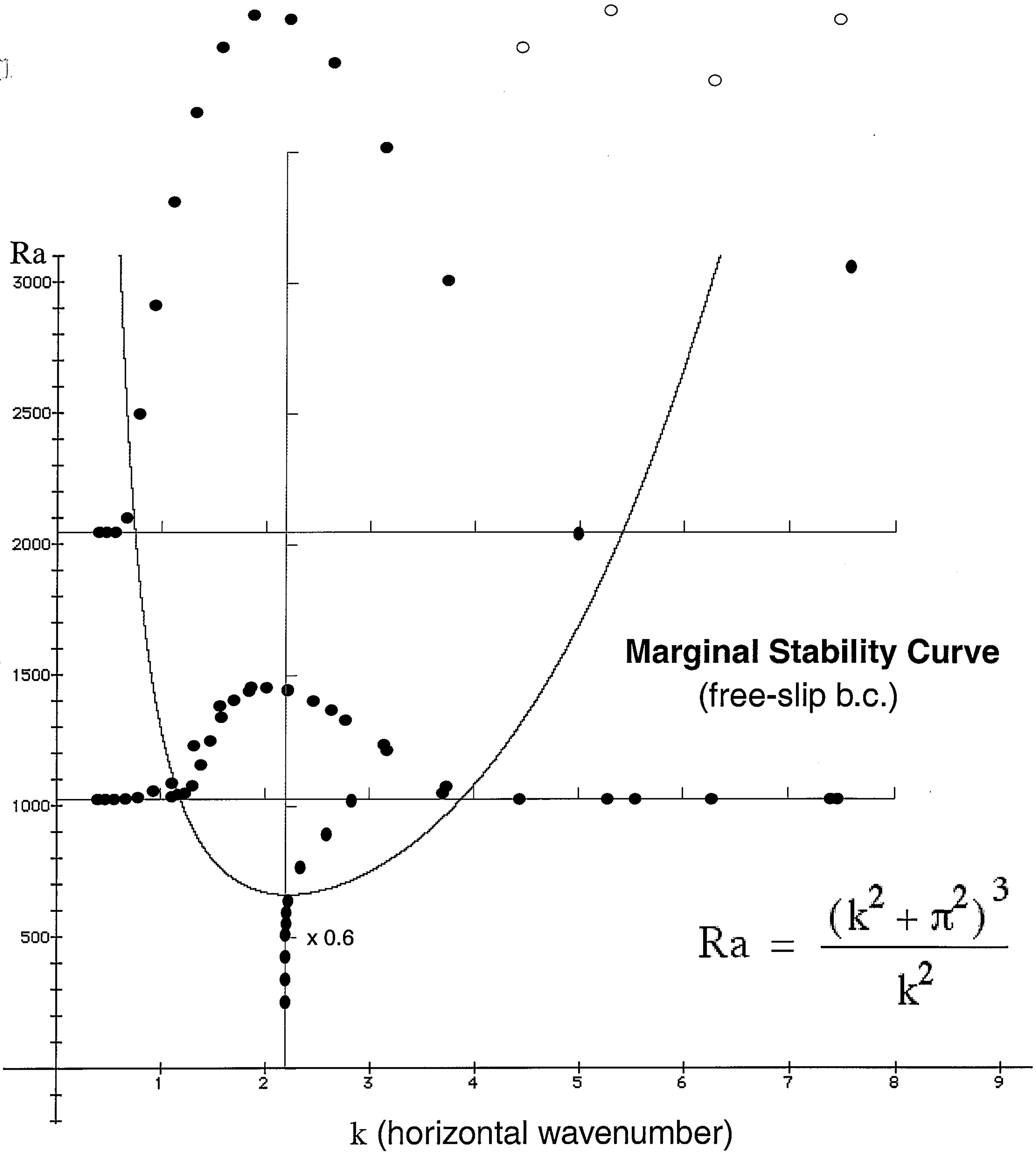
Describe linear solutions.

p. 58-59

Marginal Stability Curve
(free-slip b.c.)



$$Ra = \frac{(k^2 + \pi^2)^3}{k^2}$$



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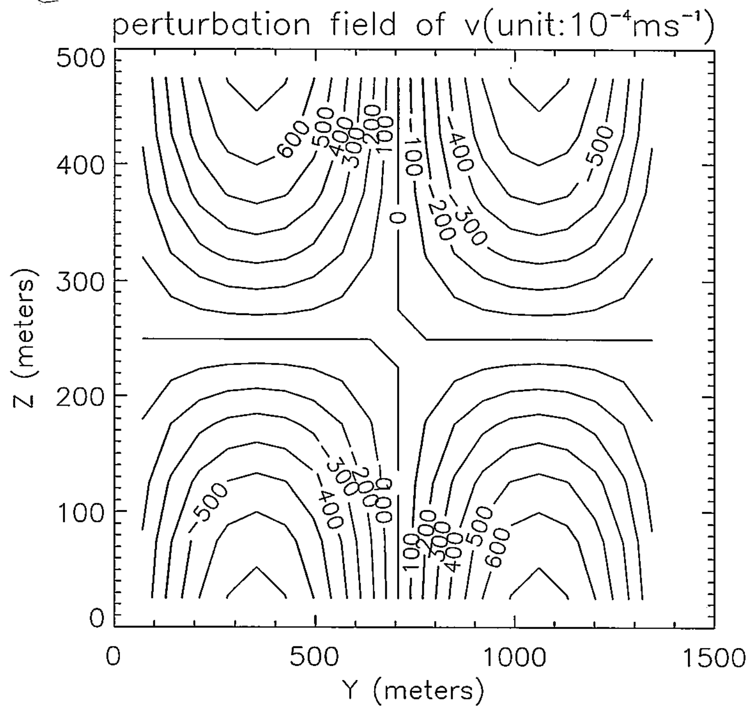
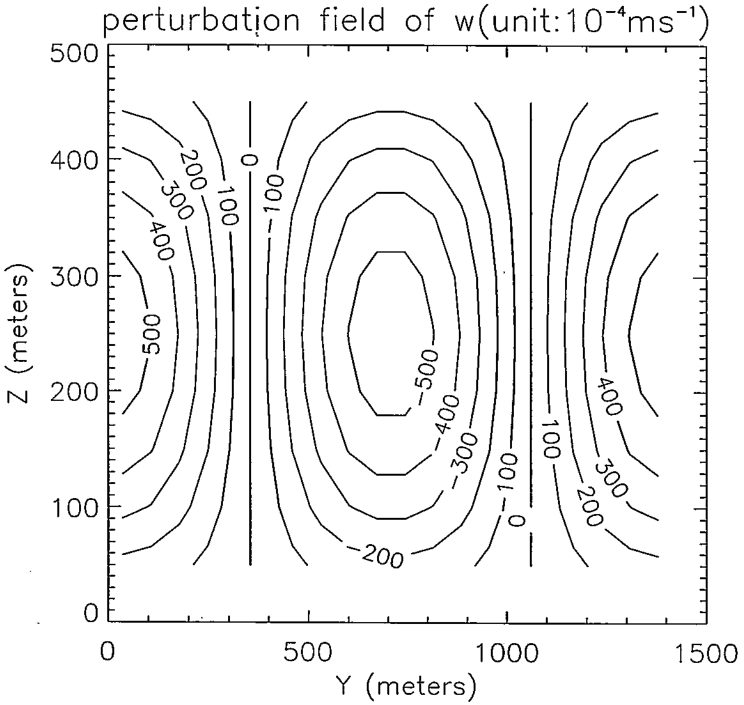
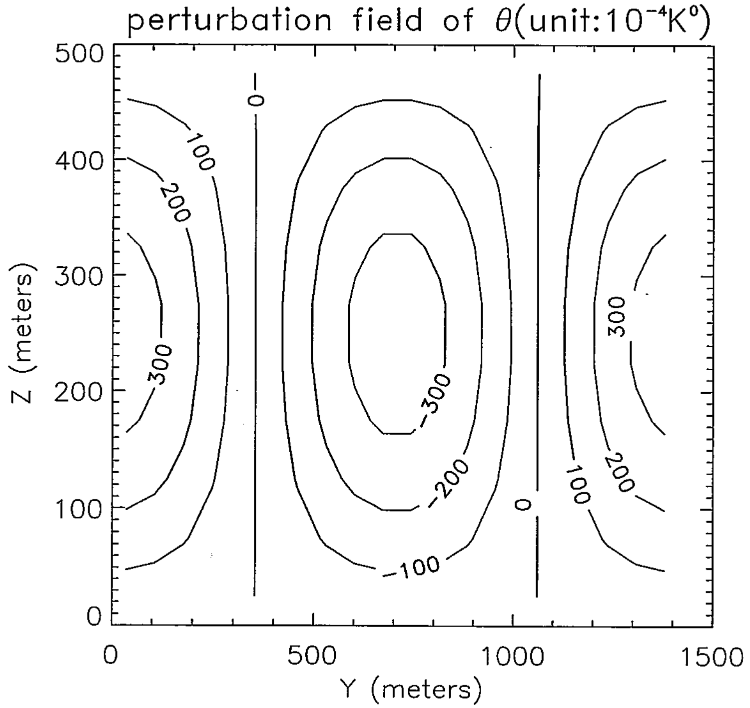
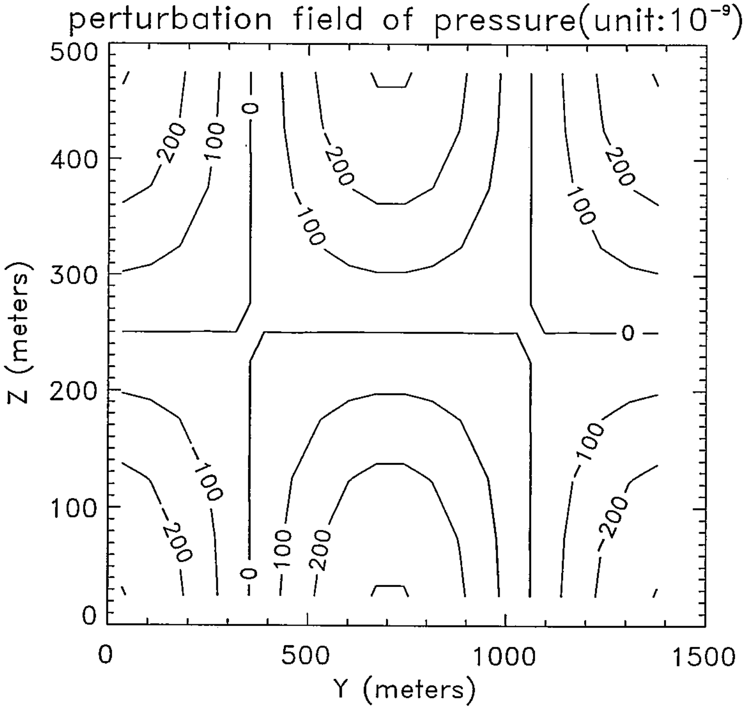

perturbation field of v(unit:$10^{-4}ms^{-1}$)


perturbation field of w(unit:$10^{-4}ms^{-1}$)


perturbation field of θ(unit:$10^{-4}K^0$)


perturbation field of pressure(unit:10^{-9})