Meteorology 6150
Shallow-layer Convection: Kinetic Energy Budget Analysis

Altocumulus cloud layers are shallow convecting layers driven by differential radiative heating. How does the intensity of the convection depend upon the mangnitude of the differential heating?

In many ways, altocumulus cloud layers resemble parallel-plate convection (PPC). In this exercise, we will use theoretical results derived earlier in the course and the results of several simulations of non-linear PPC to determine the dependence of the convection intensity upon the differential heating provided by the conductive heat flux, which we will consider to be analogous to radiative heating in an altocumulus cloud layer.

We will start by analyzing the kinetic energy equation:

$$\frac{\partial \bar{E}}{\partial t} = -\frac{\partial \overline{wE}}{\partial z} - c_p\theta_0 \frac{\partial \overline{w\pi_1}}{\partial z} + \frac{g}{\theta_0}\overline{w\theta} - \epsilon + K_v \frac{\partial^2 \bar{E}}{\partial z^2}, \tag{1}$$

where

$$\epsilon \equiv K_v \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}$$

is the (molecular) dissipation rate, and $E \equiv (v^2 + w^2)/2$. The overbar indicates a horizontal average. We have assumed that $K_w = K_v$, $\bar{v} = \bar{w} = 0$, and that the Boussinesq continuity equation,

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

applies.

Equation (1) describes the vertical distribution of the production, dissipation, and transport of $\bar{E}$. When integrated over the layer, Eq. (1) becomes

$$\frac{\partial [\bar{E}]}{\partial t} = [B] - [\epsilon], \tag{2}$$

where

$$[\bar{E}] \equiv \frac{1}{H}\int_0^H \bar{E}\, dz,$$

$$[B] \equiv \frac{1}{H}\int_0^H \frac{g}{\theta_0}\overline{w\theta}\, dz,$$

and

$$[\epsilon] \equiv \frac{1}{H} \int_0^H \epsilon dz.$$

In equilibrium, Eq. (2) becomes

$$[B] = [\epsilon]. \tag{3}$$

Equation (3) says that the buoyant generation of kinetic energy is balanced by the dissipation of kinetic energy. What does it tell us about the kinetic energy itself?

For laminar flow,

$$[\epsilon] \sim K_v \left(\frac{w_0}{H}\right)^2,$$

and

$$[B] \sim \frac{g}{\theta_0} w_0 \Delta T,$$

where w_0 is a convective velocity scale and ΔT is the temperature difference between the upper and lower boundaries.

If we use these estimates of $[\epsilon]$ and $[B]$ in (3) and solve for w_0, we obtain

$$w_0 = \frac{g}{\theta_0} \frac{\Delta T\, H^2}{K_v}. \tag{4}$$

Recall that

$$\mathrm{Ra} = \frac{g}{\theta_0} \frac{\Delta T\, H^3}{\nu \kappa}.$$

If we let $\nu = K_v$ and $\kappa = K_\theta$, then

$$\mathrm{Ra} = \frac{g}{\theta_0} \frac{\Delta T\, H^3}{K_v K_\theta}.$$

Then we can write

$$\frac{w_0}{K_\theta / H} = \mathrm{Ra}.$$

We see that the non-dimensional velocity $w_0/(K_\theta/H)$ is equal to the Rayleigh number.

We can test these theoretical predictions by using the results of several non-linear, laminar simulations. We calculate Ra and the average kinetic energy $[\bar{E}]$ for each simulation. Then we use $[\bar{E}]$ to define a convective velocity scale, $w_1 = [\bar{E}]^{1/2}$, and plot $w_1/(K_\theta/H)$ versus Ra, as shown in the attached graph.

The results do not agree with theory. This may be due to inadequate spatial resolution in the numerical model, and/or to limitations of the theory.