Meteorology 6150: Parallel-Plate Convection: Horizontally Averaged Potential Temperature and Kinetic Energy Equations

The model predicts the horizontal velocity (v), the vertical velocity (w), the potential temperature (θ), and the non-dimensional perturbation pressure (π_1). The compressible, non-rotating, adiabatic equations in Cartesian coordinates (y, z) are:

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - c_p\theta_0\frac{\partial \pi_1}{\partial y} + D_v, \tag{1}$$

$$\frac{\partial w}{\partial t} = -v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - c_p\theta_0\frac{\partial \pi_1}{\partial z} + g(\frac{\theta}{\theta_0} - 1) + D_w, \tag{2}$$

$$\frac{\partial \theta}{\partial t} = -v\frac{\partial \theta}{\partial y} - w\frac{\partial \theta}{\partial z} + D_\theta, \tag{3}$$

$$\frac{\partial \pi_1}{\partial t} = -\frac{c_s^2}{c_p\theta_0^2}\left[\frac{\partial}{\partial y}(\theta_0 v) + \frac{\partial}{\partial z}(\theta_0 w)\right]. \tag{4}$$

The terms D_v, D_w, and D_θ each have the form

$$K_\phi \nabla^2 \phi,$$

where K_ϕ is the molecular or eddy diffusivity.

The equation for the horizontally averaged potential temperature, $\bar{\theta}$ (the overbar indicates a horizontal average), is obtained from (3):

$$\frac{\partial \bar{\theta}}{\partial t} = -\frac{\partial \overline{w\theta}}{\partial z} + K_\theta\frac{\partial^2 \bar{\theta}}{\partial z^2}, \tag{5}$$

To derive (5), we assumed that the lateral boundary conditions are cyclic, and that the Boussinesq continuity equation,

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

applies. Together, these imply that $\bar{w} = 0$.

Eq. (5) can be written

$$\frac{\partial \bar{\theta}}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{w\theta} - K_\theta\frac{\partial \bar{\theta}}{\partial z}\right) = -\frac{\partial}{\partial z}\left((F_\theta)_{\mathrm{conv}} + (F_\theta)_{\mathrm{cond}}\right), \tag{6}$$

where $(F_\theta)_{\rm conv}$ and $(F_\theta)_{\rm cond}$ denote the convective and conductive vertical fluxes of potential temperature.

The equation for the horizontally averaged kinetic energy, $\bar{E}$, where $E \equiv (v^2 + w^2)/2$, is obtained from (1) and (2):

$$\frac{\partial \bar{E}}{\partial t} = -\frac{\partial \overline{wE}}{\partial z} - c_p\theta_0 \frac{\partial \overline{w\pi_1}}{\partial z} + \frac{g}{\theta_0}\overline{w\theta} - \epsilon + K_v \frac{\partial^2 \bar{E}}{\partial z^2}, \tag{7}$$

where

$$\epsilon \equiv K_v \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}$$

is the (molecular) dissipation rate. We have again assumed that the lateral boundary conditions are cyclic and that the Boussinesq continuity equation applies. In addition, we assumed that $K_w = K_v$ and $\partial\bar{v}/\partial z = 0$.

Eq. (7) can be written in a form which emphasizes the vertical fluxes of kinetic energy:

$$\begin{aligned}
\frac{\partial \bar{E}}{\partial t} &= -\frac{\partial}{\partial z}\left(\overline{wE} + c_p\theta_0\overline{w\pi_1} + -K_v\frac{\partial \bar{E}}{\partial z}\right) + \frac{g}{\theta_0}\overline{w\theta} - \epsilon, \\
&= -\frac{\partial}{\partial z}\left((F_E)_{\rm conv} + (F_E)_{\rm pres} + (F_E)_{\rm molec}\right) + B - D,
\end{aligned} \tag{8}$$

where $(F_E)_{\rm conv}$, $(F_E)_{\rm pres}$, and $(F_E)_{\rm molec}$ denote the vertical fluxes of kinetic energy due to convection, pressure-velocity correlations, and molecular processes, and B and D represent the buoyancy production and molecular dissipation of kinetic energy.