Buoyancy Oscillation

We will use the equation for the vertical acceleration of an air parcel to calculate the parcel's velocity and height as a function of time. In the following description, a variable with an overbar is a property of the environment; a variable without an overbar is a property of the parcel.

We assume that the environment of the parcel is in hydrostatic equilibrium:

$$\frac{d\bar{w}}{dt} = -g - \bar{\alpha}\frac{d\bar{p}}{dz} = 0.$$

The parcel itself will have a specific volume α and an acceleration dw/dt. We assume that the pressure of the parcel is the same as that of its environment so that

$$\frac{dw}{dt} = -g - \alpha\frac{d\bar{p}}{dz}.$$

We use the hydrostatic equation to eliminate $d\bar{p}/dz$ from this equation:

$$\frac{dw}{dt} = g\frac{\alpha - \bar{\alpha}}{\bar{\alpha}}.$$

The right hand side is called the *buoyancy* and is due to the difference in specific volume (or density) between the parcel and the environment. Substitute for α and $\bar{\alpha}$ from the equation of state for dry air, $p\alpha = RT$, (for simplicity, we assume that the air contains no water vapor) to obtain

$$\frac{dw}{dt} = g\frac{T - \bar{T}}{\bar{T}}. \tag{1}$$

Let $z = 0$ denote the parcel's equilibrium location. Then at $z = 0$, $T = \bar{T}$, and $dw/dt = 0$. Assume that the temperature in the environment varies linearly with height. Then the temperature at any height z in the environment is

$$\bar{T}(z) = \bar{T}(0) - \gamma z,$$

where $\gamma = -d\bar{T}/dz$ is the *environmental lapse rate.* Similarly, the parcel temperature at any height z is

$$T(z) = T(0) - \Gamma_d z = \bar{T}(0) - \Gamma_d z,$$

where $\Gamma_d = -dT/dz = g/c_p$ is the *dry adiabatic lapse rate.* When these expressions are substituted in Eq. (1), we obtain

$$\frac{dw}{dt} = \frac{g}{\bar{T}(0) - \gamma z}(\gamma - \Gamma_d)z \approx \frac{g}{\bar{T}(0)}(\gamma - \Gamma_d)z = bz. \tag{2}$$

Eq. (2) describes how w changes with time. By definition,

$$\frac{dz}{dt} = w. \tag{3}$$

Eqs. (2) and (3) are coupled linear differential equations which are easy to solve analytically for $z(t)$. If the coefficient b in Eq. (2) is negative, the solution $z(t)$ is sinusoidal. The parcel will oscillate about its original position with period

$$\tau = \frac{2\pi}{\sqrt{-b}} = \frac{2\pi}{\sqrt{\frac{g}{T(0)}(\Gamma_d - \gamma)}}. \tag{4}$$

If the coefficient b is positive, the solution $z(t)$ is exponentially increasing.

Numerical Solution of Ordinary Differential Equations

Eqs. (1) and (3) are examples of first-order ordinary differential equations, which have the general form

$$\frac{d\phi}{dt} = f(\phi, t). \tag{5}$$

The corresponding finite-difference form is

$$\frac{\Delta\phi}{\Delta t} \equiv \frac{\phi^{n+1} - \phi^n}{t^{n+1} - t^n} = \tilde{f}, \tag{6}$$

which can be solved for ϕ^{n+1}:

$$\phi^{n+1} = \phi^n + \tilde{f}\,\Delta t. \tag{7}$$

The superscripts n and $n+1$ indicate the time level (or time step). The tilde indicates a (to be specified) linear combination of f at time levels $n-1$, n, and $n+1$.

Many choices (schemes) are possible for $\tilde{f}$. A scheme that does not use f at time level $n+1$ is an *explicit* scheme, whereas one that does is an *implicit* scheme. Finite-difference equations that use implicit schemes often have unconditionally stable numerical solutions, which is desirable, but can also be difficult to solve, especially in coupled sets of differential equations. A scheme that uses just one time level is a *first-order* scheme, whereas one that uses two time levels is a *second-order* scheme. The *truncation error* due to finite Δt decreases $\sim \Delta t$ for first-order schemes, but $\sim (\Delta t)^2$ for second-order schemes.

Forward Euler: (explicit, first-order)

$$\tilde{f} = f^n \equiv f(\phi^n, t^n).$$

Backward Euler: (implicit, first-order)

$$\tilde{f} = f^{n+1} \equiv f(\phi^{n+1}, t^{n+1}).$$

Trapezoidal (Crank-Nicolsen): (implicit, second-order)

$$\tilde{f} = \frac{1}{2}(f^n + f^{n+1}) \equiv \frac{1}{2}\left[f(\phi^n, t^n) + f(\phi^{n+1}, t^{n+1}\right].$$

Euler Trapezoidal (Heun): (explicit, second-order)

$$\phi^* \equiv \phi^n + f(\phi^n, t^n)\,\Delta t.$$

$$\tilde{f} = \frac{1}{2}(f^n + f^*) \equiv \frac{1}{2}\left[f(\phi^n, t^n) + f(\phi^*, t^{n+1})\right].$$

Adams-Bashforth: (explicit, second-order)

$$\tilde{f} = \frac{1}{2}(3f^n - f^{n-1}) \equiv \frac{1}{2}\left[3f(\phi^n, t^n) - f(\phi^{n-1}, t^{n-1})\right].$$

The Euler Trapezoidal (Heun) scheme uses a *predictor-corrector* method. This method combines an explicit scheme (Forward Euler) with an implicit scheme (Trapezoidal) to obtain a higher-order (more accurate) yet explicit scheme.

The Adams-Bashforth scheme approximates the Trapezoidal scheme by estimating f^{n+1} (not ϕ^{n+1}) by extrapolation:

$$f^{n+1} \approx f^n + (f^n - f^{n-1}).$$

For more information on the numerical solution of ordinary differential equations, see `http://web.mit.edu/10.001/Web/Course_Notes/Differentia Equations_Notes/lec24.html`.

Numerical Solution of Buoyancy Oscillation Equations

We will solve the set of differential equations composed of Eqs. (1) and (3) which governs buoyancy oscillations in an atmosphere with any $\bar{T}(z)$:

$$\frac{dw}{dt} = g\frac{T - \bar{T}}{\bar{T}} \equiv B(z), \tag{8}$$

$$\frac{dz}{dt} = w. \tag{9}$$

To solve these numerically, we first write them in general finite-difference form, as in Eq. (7):

$$w^{n+1} = w^n + \tilde{B}\,\Delta t, \tag{10}$$

$$z^{n+1} = z^n + \tilde{w}\,\Delta t. \tag{11}$$

Next, we choose a scheme to use for $\tilde{B}$ and $\tilde{w}$. We will use the Euler Trapezoidal (Heun) scheme. Then Eqs. (10) and (11) become

$$w^* = w^n + B(z^n)\,\Delta t, \tag{12a}$$
$$z^* = z^n + w^n\,\Delta t, \tag{12b}$$

and

$$w^{n+1} = w^n + \frac{1}{2}\left[B(z^n) + B(z^*)\right]\,\Delta t, \tag{13a}$$
$$z^{n+1} = z^n + \frac{1}{2}\left(w^n + w^*\right)\,\Delta t. \tag{13b}$$

Eqs. (12a) and (12b) are the predictors, while Eqs. (13a) and (13b) are the correctors.