Analysis of fluid flow

- Uniform density: Dimensional analysis
  - Plume (space)
  - Thermal (time)
- Stably stratified fluid: Calculus, Geometry

Anelastic approx.

almost always true that the flow velocities are far less than the speed of sound, that is,

$$\frac{u_0^2}{c^2} \ll 1.$$

$$-\frac{\partial u_i'}{\partial x_i'} = \frac{u_0^2}{c^2}\left[\left(\frac{\partial \ln p}{\partial t}\right)' + u'\left(\frac{\partial \ln p}{\partial x}\right)' + v'\left(\frac{\partial \ln p}{\partial y}\right)'\right] + \frac{c_v}{c_p}\frac{D}{H}w'\left(\frac{\partial \ln p}{\partial z}\right)'. \quad (1.3.5)$$

It is therefore appropriate to neglect the first term on the right of (1.3.5). This is called the *anelastic approximation;* the resulting equation no longer contains a time derivative and is therefore a *diagnostic equation*[1] which

Boussinesq approx

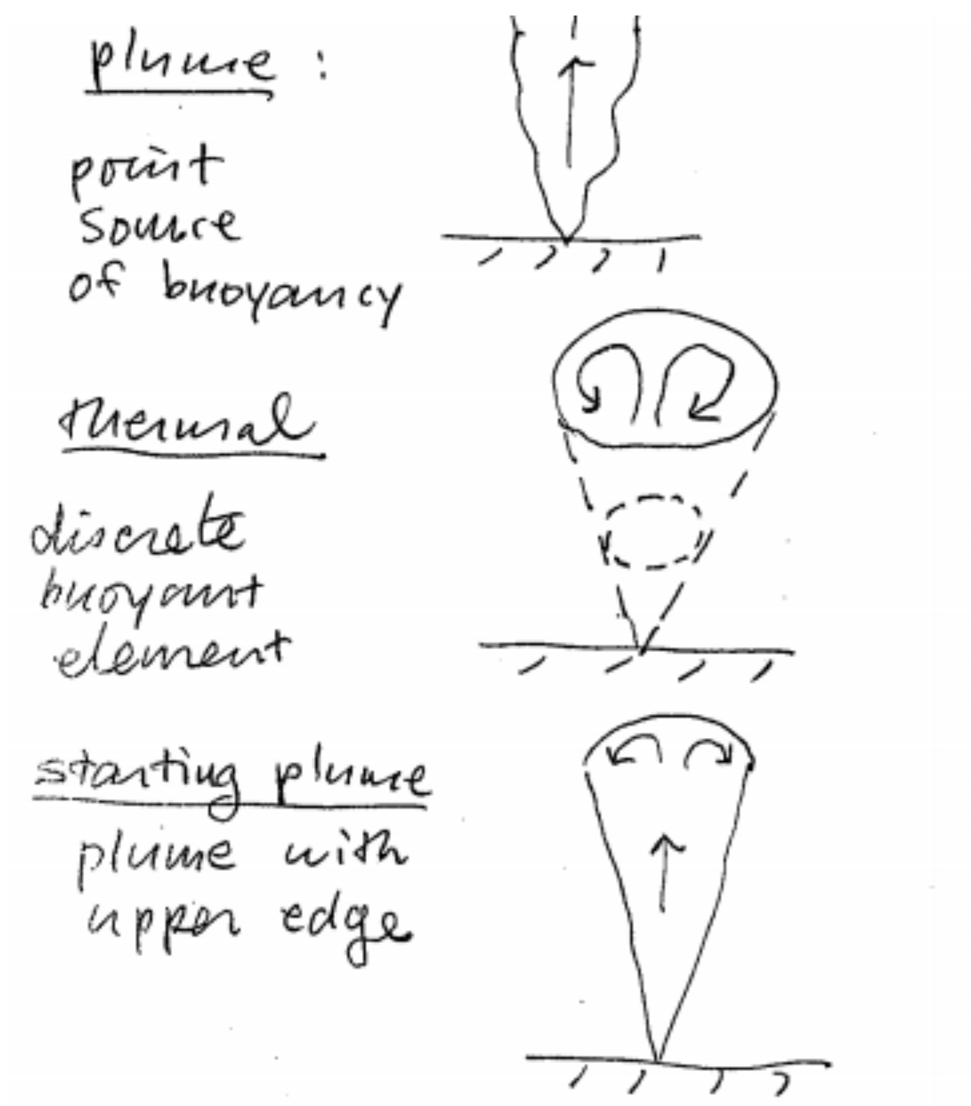
If it is also true that the depth through which the convective motion occurs is much less than the scale height (about 10 km

the Navier-Stokes equations may be written:

$$(\overline{\rho} + \rho')\,\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} - (\overline{\rho} + \rho')\,f_i + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij}\right], \quad (1.3.7)$$

der that the system be energetically consistent. Therefore, *the Boussinesq approximation neglects density variations in the fluid except when they are coupled with gravity* [f_i in (1.3.7)].

Local convection



Local convection

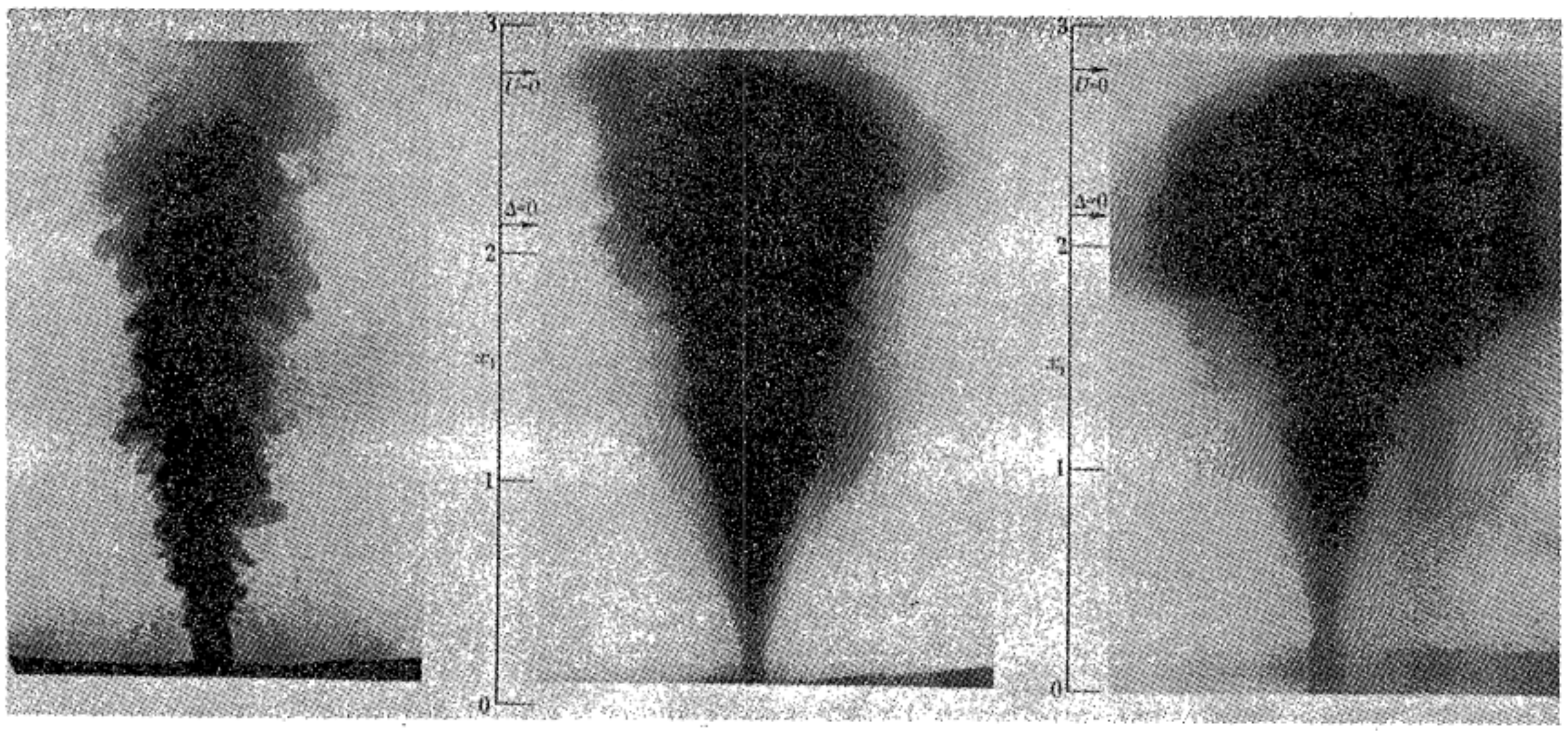


Fig. 2.9 Photographs of plumes in neutrally and stably stratified fluids. At left is a plume in a neutrally stratified ambient fluid; at right are time exposures of a plume in a stable stratified fluid at early and late stages in its development. [*From Morton, Taylor, and Turner (1956).*]

Local convection

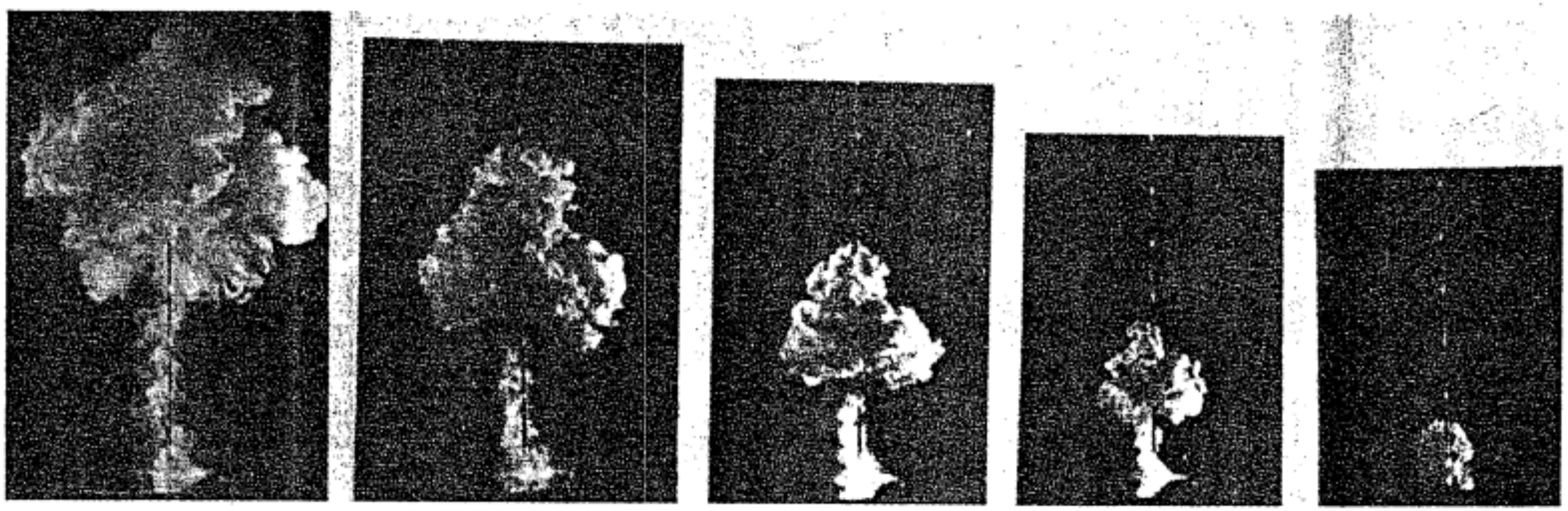


Fig. 2.14 Successive photographs of a descending thermal, showing that the shape of the thermal may persist while the volume increases several times [*From Scorer (1957).*]

Plume Case

- Assumptions

the flow is fully turbulent, then it should be independent of the magnitude of the molecular diffusivities. If the Boussinesq approximation is applicable, then the *only* relevant dimensional parameter in the problem is the rate F at which buoyancy is supplied by the point source! (As the source is regarded as a point, it has no dimensions associated with it.) As the flow is driven by buoyancy, there are no other fluid properties that are relevant to this problem.

The buoyancy flux F has the dimensions of

$$F \sim \text{Buoyancy} \times \text{Velocity} \times \text{Area} = L^4 t^{-3}, \qquad (2.2.1)$$

Buckingham Pi theorem

If the equation $\varphi(q_1, q_2, q_3, \ldots, q_n) = 0$ is the only relationship among the n q's and if it holds for any arbitrary choice of units in which $q_1, q_2, q_3, \ldots, q_n$ are measured, then the relation $\varphi(\pi_1, \pi_2, \pi_3, \ldots, \pi_m) = 0$ is satisfied where $\pi_1, \pi_2, \ldots, \pi_m$ are independent dimensionless products of the q's. Furthermore, if k is the minimum number of primary quantities necessary to express the dimensions of the q's, then

$$m = n - k.$$

Plume

- Properties depend only on F and z

$$\overline{w} = c_1 f(F, z)$$
$$\overline{B} = c_2 g(F, z)$$

- Derive on board

Result: $\overline{w} = c_1 F^{1/3} z^{-1/3}$

similarly: $\overline{B} = c_2 F^{2/3} z^{-5/3}$.

Plume

- What should the equation be for the mean radius of the plume?

Plume

What about structure of plume?
Must depend on r/R (must be dimensionless)!

$$w = \frac{F^{1/3}}{z^{1/3}} \times \text{func}(r/R)$$

$$B = \frac{F^{2/3}}{z^{5/3}} \times \text{func}(r/R)$$

$$R = \alpha z$$

$$w = 4.7\frac{F^{1/3}}{z^{1/3}}\exp\left(\frac{-96r^2}{z^2}\right),$$

$$B = 11.0\frac{F^{2/3}}{z^{5/3}}\exp\left(\frac{-71r^2}{z^2}\right),$$

$$R = 0.12z.$$

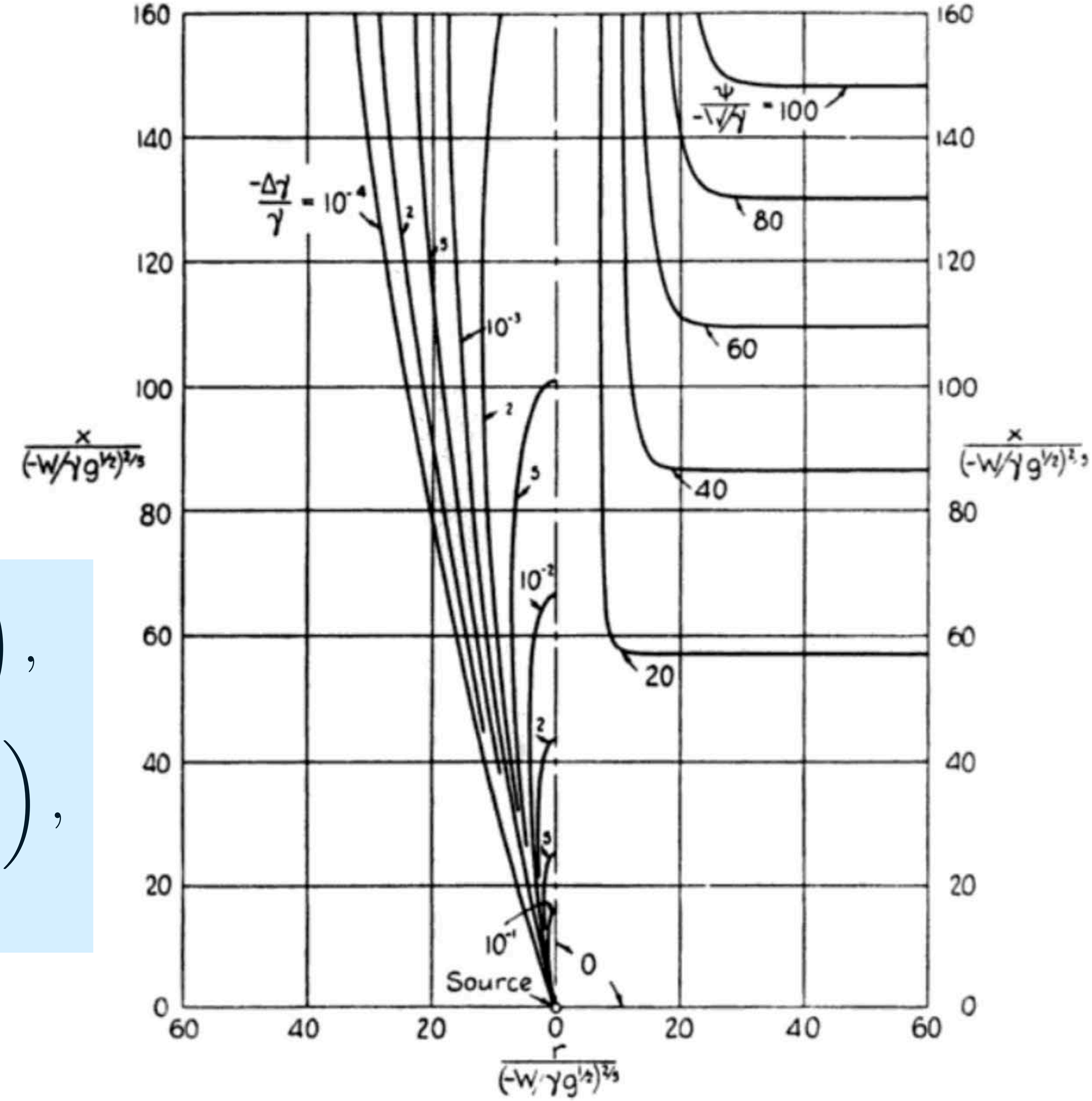


Fig. 2.2 Mean isotherms and streamlines for the turbulent convection due to a maintained point source. The isotherms are labeled with the values of $(T - T_0)/T$, while the streamlines are labeled with relative values of the Stokes stream function. [*(After Rouse, Yih, and Humphreys (1952).*]

Plume mass flux

- Mass flux proportional to W times Area
- Derive on board

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$$\overline{w} R^2 = \frac{F^{1/3}}{z^{1/3}} \alpha^2 z^2 \sim z^{5/3}$$

so it increases with z. Implies entrainment. mean inflow velocity must be linearly proportional to w (by dimensional analysis).

Line source of convection

- Similar analysis, except now F has units of buoyancy flux per unit length (along the source)

Thermals

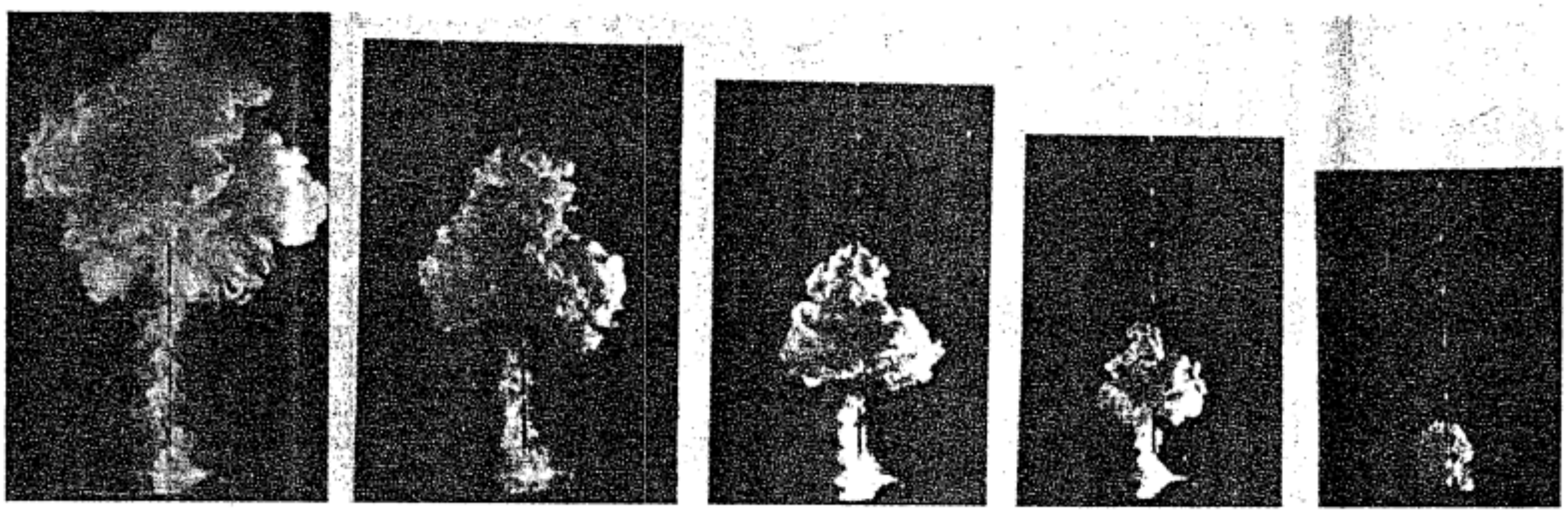
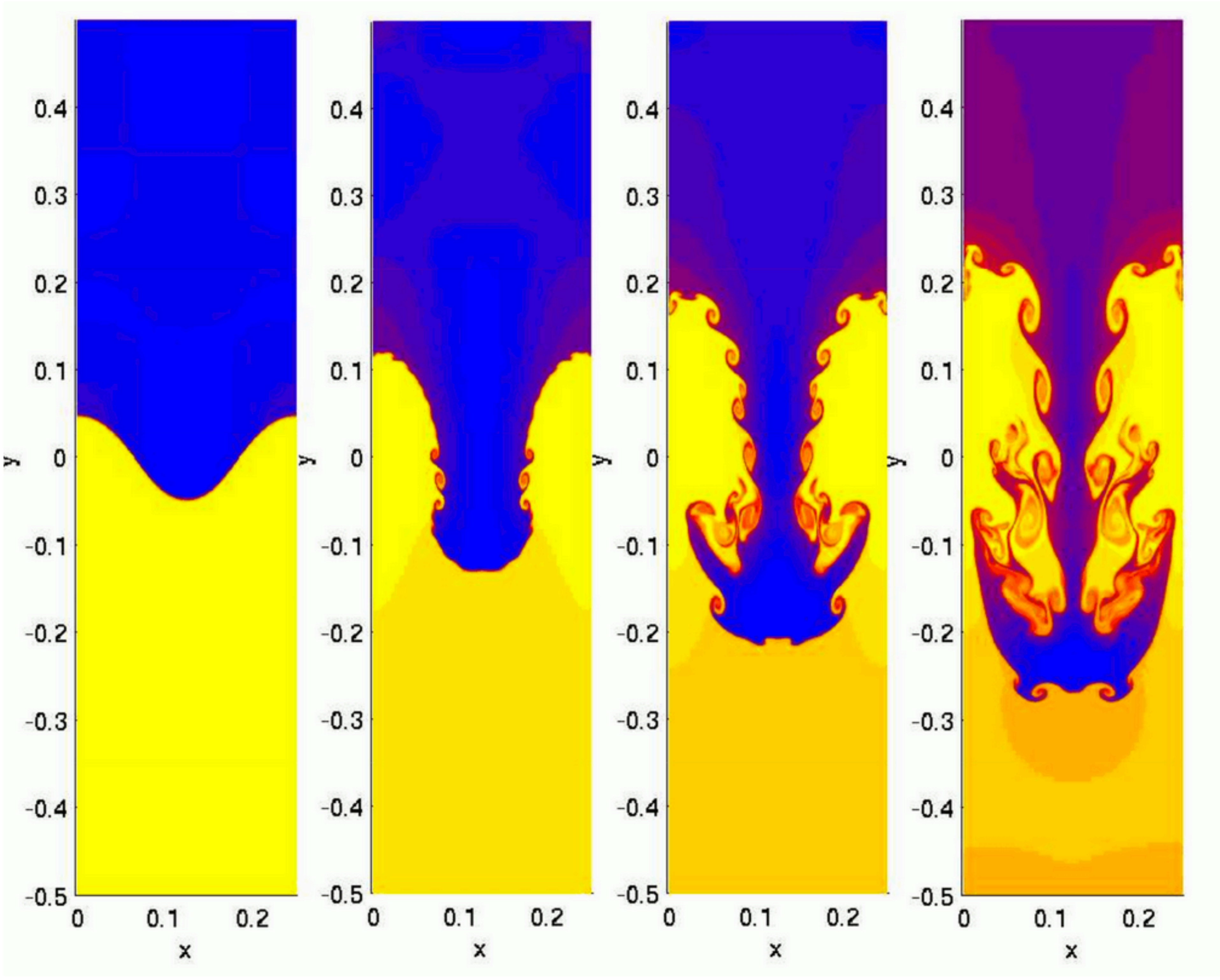


Fig. 2.14 Successive photographs of a descending thermal, showing that the shape of the thermal may persist while the volume increases several times [*From Scorer (1957).*]

Thermals



Thermals

- Same as plumes, but now regard time as the key variable rather than height
- The same assumptions for the plume still apply (self-similarity and Boussinesq)

Thermals

In a <u>neutrally stratified fluid</u>, only external parameter is amount of buoyancy released at source:

$$Q \equiv \iiint B_0 \, d\tau \qquad \text{(vol. integral)}$$

Thermals

From dimensional analysis, with z referring to ht. of "center" of thermal at time t:

$$w = \frac{Q^{1/2}}{z} \times \text{func}\left(\frac{r}{R}\right)$$

$$B = \frac{Q}{z^3} \times \text{func}\left(\frac{r}{R}\right)$$

$$R = \gamma z .$$ mean radius

Check:

$$[Q] = [B]\,L^3 = \frac{L}{t^2}\,L^3 = L^4 t^{-2}$$

$$[w] = \frac{L}{t}$$

$$[Q^{1/2} z^{-1}] = L^2 t^{-1} L^{-1} = L t^{-1} = [w]$$

Thermals

- How does z relate to t?

Use dimensional analysis!

$z = c\,t^{a} Q^{b}$ (c: dimensionless constant)

$[z] = [t]^{a}[Q]^{b}$

$L = t^{a} L^{4b} t^{-2b}$

$L: \quad 1 = 4b \rightarrow b = 1/4$

$t: \quad 0 = a - 2b \rightarrow a = 1/2$

$$\boxed{z = c\,t^{1/2} Q^{1/4}}$$

Thermals

- How does z relate to t?

$$z = c\, t^{1/2} Q^{1/4}$$

Then it follows that

$$w \sim t^{-1/2}$$
$$B \sim t^{-3/2}$$

Thermals

- Plot of z^2 vs t with thermal outline

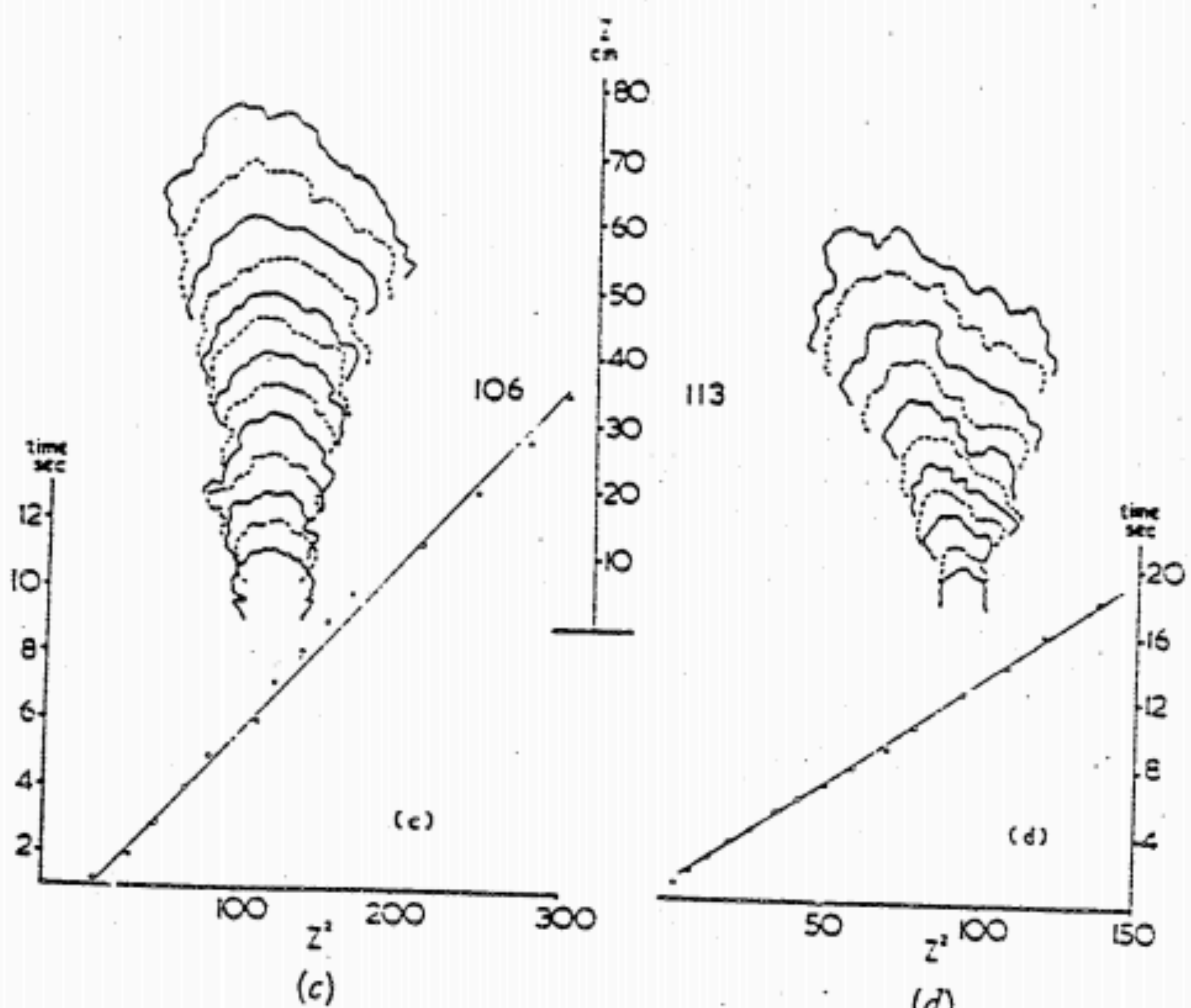


Fig. 2.15 Successive outlines of thermals traced from photographs. Below each is a graph of z^2 against t. [*From Scorer (1957).*]

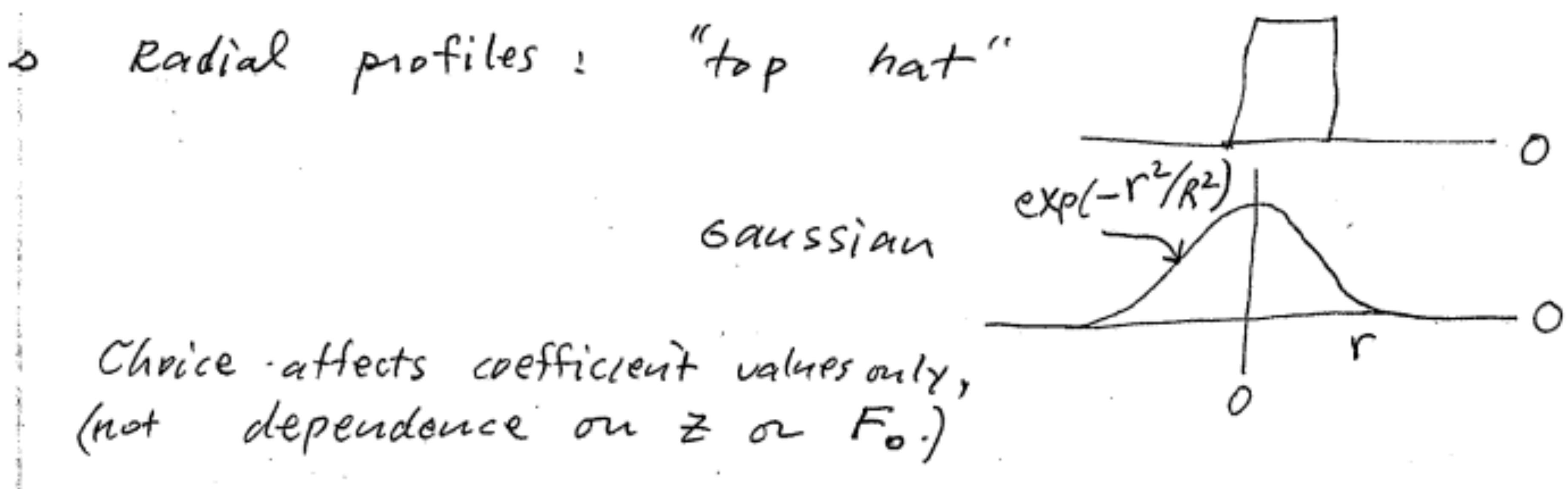
Turbulent convection in stably stratified fluid

- Stable means density increases with height
- Even without entrainment, positive buoyancy is reduced as the buoyant element moves upward

Turbulent convection in stably stratified fluid

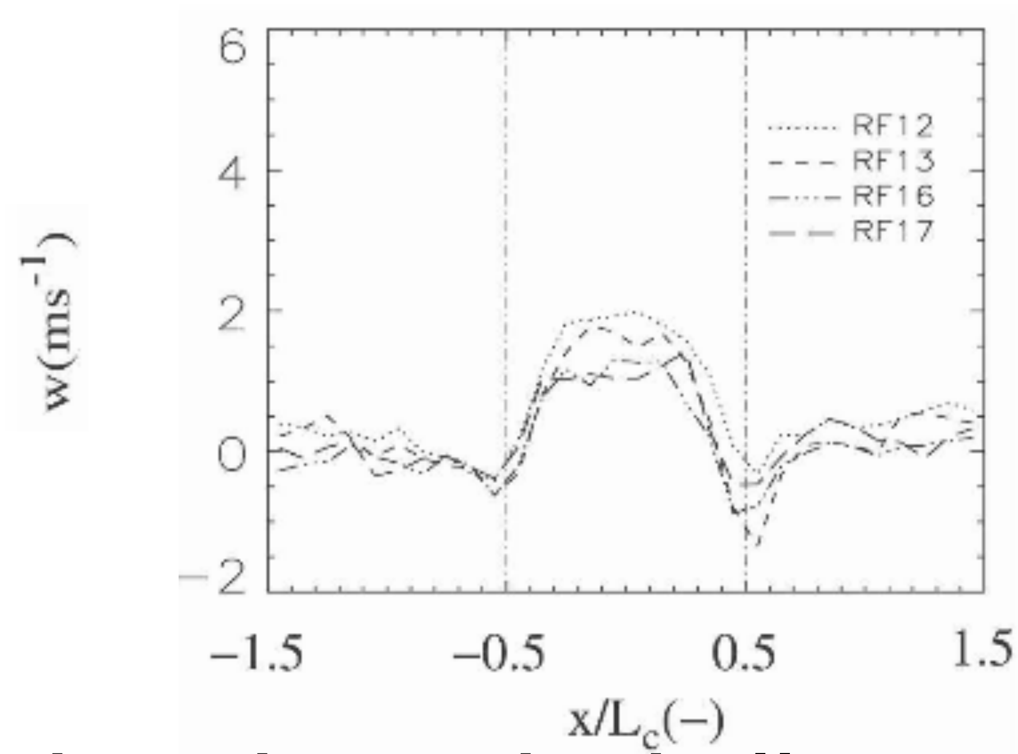
- Adding an additional parameter (stability)
- Use the actual governing equations (Boussinesq)
- Must assume a geometric structure

Turbulent convection in stably stratified fluid



we assume a top-hat profile.

Top hat profile



- Flights through shallow cumulus (Rodts et al. 2003 (JAS)

The primary assumptions made in the course of solving the governing equations are borrowed from the self-similar solutions in unstratified flow:

1) The flow is steady.
2) The radial profiles of mean vertical velocity and mean buoyancy are similar at all heights.
3) The mean turbulent inflow velocity is proportional to vertical velocity.
4) The flow is Boussinesq.

Continuity Equation

$\nabla \cdot \underset{\sim}{v} = 0$

in radial (cylindrical) coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z} w = 0 \qquad u = \frac{dr}{dt}$$

Integrate over area of plume:

Continuity Equation

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$$\int_0^{2\pi}\int_0^R \frac{1}{r}\frac{\partial}{\partial r}(ru)\, r\, dr\, d\theta$$

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A

$$+ \frac{\partial}{\partial z}\int_0^{2\pi}\int_0^R w\, r\, dr\, d\theta = 0.$$

B

$$A = 2\pi \int_0^R \frac{\partial}{\partial r}(ru)\, dr = 2\pi (ru)\mid_0^R = 2\pi R u(R) = -2\pi\alpha R w$$

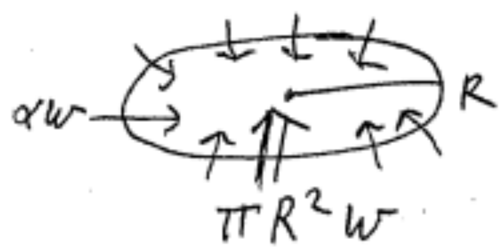
$$B = \frac{\partial}{\partial z}\left(2\pi w \int_0^R r\, dr\right) = \frac{\partial}{\partial z}\left(\pi R^2 w\right)$$

Continuity Equation

use $u = -\alpha w$: (entrainment relation).

$$2\pi R \alpha w = \frac{d}{dz}\left(\pi w R^2 \right) \qquad (2.7.1)$$

↑ entrainment ↑ mass flux



mass flux increases due to entrainment.

Vertical Momentum

(b) <u>vertical momentum eq.</u> (steady)
(neglect pert. p.g.f.)

$$\frac{dw}{dt} = \nabla \cdot (\underline{V} w) = B$$

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$\hat{n}$: unit normal vector to surface (S).

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By definition, $w = 0$ at $r = R$, so we get

$$\underset{\text{(top)}}{\left[\pi R^2 w^2 + \frac{d}{dz}(\pi R^2 w^2)\Delta z\right]} - \underset{\text{(bottom)}}{\pi R^2 w^2} = B \pi R^2 \Delta z$$

$$\boxed{\frac{d}{dz}(\pi R^2 w^2) = \pi R^2 B} \quad (2.7.2.)$$

this says that increase of vertical kinetic energy (w^2) is due to buoyancy.

Thermodynamic Eq'n

First Law is $c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} = 0$.

Definè $\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$. Then $\frac{d\theta}{dt} = 0$.

Assume steady,

$$\frac{d\theta}{dt} = 0 = \nabla\cdot(\underset{\sim}{V}\theta) = \nabla\cdot\underset{\sim}{V}(\theta - \theta_0) \equiv \nabla\cdot\underset{\sim}{V}B'$$

let $\left\{\begin{array}{l}\theta = \text{plume pot. temp.}\\ \bar{\theta} = \text{ambient fluid}\end{array}\right\}$

θ_0 = constant ref. pot. temp.

Thermodynamic Eq'n

Again use div. theorem:

$$\iiint \nabla \cdot \underset{\sim}{V} B' \, d\tau = \oint B' \underset{\sim}{V} \cdot \hat{n} \, dS = 0.$$

Evaluate surface integral over surface of incremental volume:

$$\left\{ (\theta - \theta_0) \pi R^2 w + \frac{d}{dz}\left[(\theta - \theta_0) \pi R^2 w \right] \Delta z \right\} \quad \text{(top)}$$

$$- (\theta - \theta_0) \pi R^2 w \quad \text{(bottom)} \quad - 2\pi R \Delta z \, q w \, (\bar{\theta} - \theta_0) = 0 \quad \text{(sides)}$$

ambient

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($\bar{\theta}$ ← ambient)

$$\frac{d}{dz}\left[\pi R^2 w (\theta - \theta_0)\right] = 2\pi R \alpha w \left(\bar{\theta} - \theta_0\right). \qquad (2.7.3)$$

$$\frac{d}{dz}\left[\pi R^2 w\,(\theta - \theta_0)\right] = 2\pi R\alpha w\left(\overline{\theta} - \theta_0\right). \tag{2.7.3}$$

Since, from (2.7.1),

$$2\pi\alpha R w = \frac{d}{dz}(\pi R^2 w),$$

(2.7.3) may be rewritten:

$$\frac{d}{dz}\left[\pi R^2 w\,(\theta - \theta_0)\right] = \left(\overline{\theta} - \theta_0\right)\frac{d}{dz}\left(\pi R^2 w\right)$$

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or

$$\frac{d}{dz}\left[\pi R^2 w\left(\theta-\overline{\theta}\right)\right]=-\pi R^2 w\frac{d\overline{\theta}}{dz}.$$

The above is multiplied through by g/θ_0 and we arrive at

$$\boxed{\frac{d}{dz}\left(\pi R^2 w B\right) = -\pi R^2 w N^2,} \tag{2.7.4}$$

where

$$N^2 \equiv \frac{g}{\theta_0}\frac{d\overline{\theta}}{dz}.$$

N has the dimensions of $(\text{time})^{-1}$ and is called the Brunt-Väisälä or buoyancy frequency. In a stably stratified fluid, N is the frequency at which an infinitesimal sample of fluid oscillates if displaced vertically.

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this eq. says that the change in buoyancy (or heat) flux is due to vertical motion in a mean gradient. If $N^2 = 0$, then $\pi R^2 w B$ = constant = F, the boundary buoyancy flux.

Equation Summary

Summary:

Mass flux: $$\frac{d}{dz}(R^2 w) = 2 R \alpha w \qquad (2.7.5)$$

Momentum flux ($\propto$ K.E.): $$\frac{d}{dz}(R^2 w^2) = R^2 B \qquad (2.7.6)$$

Heat ($\propto$ buoyancy) flux: $$\frac{d}{dz}(R^2 w B) = - R^2 w N^2 \qquad (2.7.7)$$