2.1 *Effects of a cumulus ensemble upon the large–scale temperature and moisture fields by induced subsidence and detrainment*

Arakawa (1969, 1972), W. Gray (1971), Ooyama (1971), Pearce and Riehl (1969), and Yanai (1971) were among the first to recognize the importance of cumulus–induced subsidence and detrainment of cloud air in modifying the large–scale environment. Arakawa (1969) proposed a parameterization scheme for a three-layer general circulation model, in which the role of mass flux in a cloud ensemble and detrainment from the clouds were explicitly included, but only a single cloud type was allowed to occur at a given time and space location. Arakawa (1972) wrote the scheme in a more general continuous form, but the single cloud–type assumption was still retained. Yanai (1971) proposed an observational program to obtain the cloud mass flux, combining the large–scale heat and moisture budgets with this cloud model.

(a) *Preliminary considerations*

For convenience, we define the *dry static energy* as

$$s \equiv c_p T + gz, \tag{1}$$

and the *moist static energy* as

$$h \equiv c_p T + gz + Lq, \tag{2}$$

where s is the sum of enthalpy and potential energy, and h the sum of s and latent energy. For dry adiabatic processes,

$$\frac{ds}{dt} \approx 0, \tag{3}$$

while for both dry and moist adiabatic processes,

$$\frac{dh}{dt} \approx 0. \tag{4}$$

We consider an ensemble of cumulus clouds that is embedded in a tropical large-scale motion system. The thermodynamic energy equation is

$$\frac{ds}{dt} = Q_R + L(c - e), \tag{5}$$

where Q_R is the radiative heating rate, c the rate of condensation, e the rate of evaporation of cloud condensate per unit mass of air. The continuity equation for moisture is

$$\frac{dq}{dt} = e - c. \tag{6}$$

With the aid of the (anelastic) continuity equation,

$$\nabla_h \cdot (\rho \vec{\mathbf{V}}_{\mathbf{h}}) + \frac{\partial(\rho\, w)}{\partial z} = 0, \tag{7}$$

(5) and (6) are written in the flux forms

$$\frac{\partial(\rho\, s)}{\partial t} + \nabla_h \cdot (\rho\, s\, \vec{\mathbf{V}}_{\mathbf{h}}) + \frac{\partial(s\, m)}{\partial z} = \rho\, Q_R + \rho\, L(c - e), \tag{8}$$

$$\frac{\partial(\rho\, q)}{\partial t} + \nabla_h \cdot (\rho\, q\, \vec{\mathbf{V}}_{\mathbf{h}}) + \frac{\partial(q\, m)}{\partial z} = \rho\, (e - c), \tag{9}$$

where $m \equiv \rho\, w$ is the <u>vertical mass flux</u> per unit area.

We consider a horizontal area that is large enough to contain an ensemble of cumulus clouds but small enough to be regarded as a fraction of a large–scale motion system. Taking horizontal averages of (8) and (9), we obtain

$$\frac{\partial(\rho\, \overline{s})}{\partial t} + \nabla_h \cdot \overline{(\rho\, s \vec{\mathbf{V}}_{\mathbf{h}})} + \frac{\partial(\overline{s}\, \overline{m})}{\partial z} = \rho\, \overline{Q}_R + \rho\, L(\, \overline{c - e}\,) - \frac{\partial(\overline{s' m'})}{\partial z}, \tag{10}$$

$$\frac{\partial(\rho\, \overline{q})}{\partial t} + \nabla_h \cdot \overline{(\rho\, q \vec{\mathbf{V}}_{\mathbf{h}})} + \frac{\partial(\overline{q}\, \overline{m})}{\partial z} = \rho(\, \overline{e - c}\,) - \frac{\partial(\overline{q' m'})}{\partial z}, \tag{11}$$

where the overbar $(^-)$ is the horizontal average, and $(\ )'$ is the deviation from the average.

Let σ be the fractional area occupied by active cumulus clouds. ($\sigma = \sum_i \sigma_i$.) For simplicity, we assume that *all clouds have the same characteristic values of s, q, and $\rho\, w$ at a given level.* Then

$$\begin{aligned} \overline{s} &= \sigma\, s_c + (1 - \sigma)\widetilde{s}, \\ \overline{q} &= \sigma\, q_c + (1 - \sigma)\widetilde{q}, \end{aligned} \tag{12}$$

$$\begin{aligned} \overline{m} \equiv \overline{\rho\, w} &= \underbrace{\sigma\, m_c} + \underbrace{(1 - \sigma)\widetilde{m}}, \\ &= \quad M_c + \quad \widetilde{M}. \end{aligned} \tag{13}$$

M_c is called the cloud mass flux, and $\widetilde{(\ )}$ is the value in the cloud–free environment.

$$\overline{s\,m} = \sigma\, s_c\, m_c + (1-\sigma)\widetilde{s}\,\widetilde{m},$$
$$\overline{s}\,\overline{m} = \sigma^2\, s_c\, m_c + (1-\sigma)^2\, \widetilde{s}\,\widetilde{m} + \sigma(1-\sigma)(\widetilde{s}\, m_c + s_c\,\widetilde{m}).$$

Therefore

$$\begin{aligned}\overline{s'\,m'} &= \overline{s\,m} - \overline{s}\,\overline{m} \\ &= \sigma(1-\sigma)(s_c - \widetilde{s})(m_c - \widetilde{m}).\end{aligned} \tag{14}$$

• We will make use of the facts:

$$\sigma << 1 \tag{15a}$$

$$|s_c - \widetilde{s}| << \widetilde{s}, \quad |q_c - \widetilde{q}| << \widetilde{q}. \tag{15b}$$

[(15b) are strong (sufficient) conditions. They are not necessary.] Then we obtain

$$\boxed{\begin{aligned}\overline{s} &= \sigma(s_c - \widetilde{s}) + \widetilde{s} \approx \widetilde{s}, \\ \overline{q} &= \sigma(q_c - \widetilde{q}) + \widetilde{q} \approx \widetilde{q},\end{aligned} \qquad (16)}$$

However, $\overline{m}$ (mean large–scale vertical mass flux) $\neq \widetilde{m}$ (vertical mass flux of the cloud environment).

(Case 1):

$$\overline{m} = 0 \quad \text{(no net vertical mass flux)}$$
$$\widetilde{m} = -\frac{\sigma}{1-\sigma} m_c \quad \text{(compensating downward mass flux)}$$

Therefore

$$m_c - \widetilde{m} = m_c\Big(1 + \frac{\sigma}{1-\sigma}\Big) = \frac{m_c}{1-\sigma}.$$

and (exact)

$$\overline{s'm'} = \sigma\, m_c(s_c - \widetilde{s}) = M_c(s_c - \widetilde{s}). \tag{17a}$$

(Case 2):

$$\widetilde{m} = 0 \quad \text{(all ascent is in clouds only)}$$
$$\overline{s'm'} = \sigma\, m_c(1-\sigma)(s_c - \widetilde{s}) \approx M_c(s_c - \widetilde{s}). \tag{17b}$$

Therefore in general we have

$$\overline{s'm'} \approx \sigma m_c(s_c - \widetilde{s}) = M_c(s_c - \widetilde{s}). \tag{18}$$

Therefore (10) and (11) become

$$\frac{\partial(\rho\,\overline{s})}{\partial t}+\nabla_h\cdot\overline{(\rho\,s\vec{\mathbf{V}}_{\mathbf{h}})}+\frac{\partial(\overline{s}\,\overline{m})}{\partial z}=\underbrace{\rho\,Q_R+\rho\,L(c-e)-\frac{\partial}{\partial z}[M_c(s_c-\widetilde{s})]}_{apparent\ heat\ source},\tag{19}$$

$$\frac{\partial(\rho\,\overline{q})}{\partial t}+\nabla_h\cdot\overline{(\rho\,q\vec{\mathbf{V}}_{\mathbf{h}})}+\frac{\partial(\overline{q}\,\overline{m})}{\partial z}=\underbrace{\rho(e-c)-\frac{\partial}{\partial z}[M_c(q_c-\widetilde{q})]}_{apparent\ moisture\ source},\tag{20}$$

where the $(^{-})$ symbol has been omitted for Q_R, c, and e. From (19) and (20) we have also

$$\frac{\partial(\rho\,\overline{h})}{\partial t}+\nabla_h\cdot\overline{(\rho\,h\vec{\mathbf{V}}_{\mathbf{h}})}+\frac{\partial(\overline{h}\,\overline{m})}{\partial z}=\rho\,Q_R-\frac{\partial}{\partial z}[M_c(h_c-\widetilde{h})],\tag{21}$$

where $h=s+Lq$ (moist static energy).

(b) *The effects of cumulus–induced subsidence and detrainment*

So far we have expressed the vertical cumulus transports of sensible heat and moisture in terms of the cloud mass flux M_c and the excess of s_c (thus T_c) and q_c from the environmental values.

For simplicity let us ignore the radiative heating Q_R and evaporation of clouds e. Then (19) and (20) are simplified as

$$\frac{\partial(\rho\,\overline{s})}{\partial t}+\nabla_h\cdot\overline{(\rho\,s\vec{\mathbf{V}}_{\mathbf{h}})}+\frac{\partial(\overline{s}\,\overline{m})}{\partial z}=\rho\,Lc-\frac{\partial}{\partial z}[M_c(s_c-\widetilde{s})],\tag{22}$$

$$\frac{\partial(\rho\,\overline{q})}{\partial t}+\nabla_h\cdot\overline{(\rho\,q\vec{\mathbf{V}}_{\mathbf{h}})}+\frac{\partial(\overline{q}\,\overline{m})}{\partial z}=-\rho c-\frac{\partial}{\partial z}[M_c(q_c-\widetilde{q})],\tag{23}$$

We assume that *the cumulus ensemslbe maintains a heat and moisture balance with the large–scale environment. Thus there will be no accumulation of heat or moisture in the cumulus ensemble over a time scale describing the large–scale motion.* The balance equations for the cumulus ensemble are

$$\frac{\partial(\rho\,\sigma s_c)}{\partial t}=\frac{\partial M_c}{\partial z}\widetilde{s}-\frac{\partial(M_c\,s_c)}{\partial z}+\rho\,L\,c=0,\qquad\text{(heat balance)}\tag{24}$$

$$\frac{\partial(\rho\,\sigma q_c)}{\partial t}=\frac{\partial M_c}{\partial z}\widetilde{q}-\frac{\partial(M_c\,q_c)}{\partial z}-\rho\,c=0,\qquad\text{(moisture balance)}\tag{25}$$

$$\frac{\partial(\rho\,\sigma h_c)}{\partial t} = \underbrace{\frac{\partial M_c}{\partial z}\widetilde{h}}_{entrainment} - \frac{\partial(M_c\,h_c)}{\partial z} = 0, \qquad \text{(moist static energy balance)} \tag{26}$$

for the entrainment layer where $\partial M_c/\partial z > 0$, and

$$\frac{\partial(\rho\,\sigma s_c)}{\partial t} = \frac{\partial M_c}{\partial z} s_c - \frac{\partial(M_c\,s_c)}{\partial z} + \rho\,L\,c = 0, \tag{27}$$

$$\frac{\partial(\rho\,\sigma q_c)}{\partial t} = \frac{\partial M_c}{\partial z} q_c - \frac{\partial(M_c\,q_c)}{\partial z} - \rho\,c = 0, \tag{28}$$

$$\frac{\partial(\rho\,\sigma h_c)}{\partial t} = \underbrace{\frac{\partial M_c}{\partial z} h_c}_{entrainment} - \frac{\partial(M_c\,h_c)}{\partial z} = 0, \tag{29}$$

for the detrainment layer where $\partial M_c/\partial z < 0$,

- For the entrainment layer, we substitute (25) into the right–hand side of (22) to find

$$\begin{aligned}
\rho\,L\,c - \frac{\partial}{\partial z}[M_c(s_c - \widetilde{s})] &= L\left[\frac{\partial M_c}{\partial z}\widetilde{q} - \frac{\partial(M_c\,q_c)}{\partial z}\right] - \frac{\partial}{\partial z}\Big[M_c(s_c - \widetilde{s})\Big] \\
&= \frac{\partial M_c}{\partial z} L\,\widetilde{q} - \frac{\partial(M_c\,h_c)}{\partial z} + \frac{\partial(M_c\,\widetilde{s})}{\partial z} \\
&= \underbrace{\frac{\partial M_c}{\partial z}\widetilde{h} - \frac{\partial(M_c\,h_c)}{\partial z}}_{=0\ by\ (26)} + M_c\frac{\partial \widetilde{s}}{\partial z}
\end{aligned} \tag{30}$$

Therefore (22) becomes

$$\frac{\partial(\rho\,\overline{s})}{\partial t} + \nabla_h \cdot \overline{(\rho\,s\vec{\mathbf{V}}_{\mathbf{h}})} + \frac{\partial(\overline{s}\,\overline{m})}{\partial z} = M_c\frac{\partial\widetilde{s}}{\partial z}. \tag{31}$$

Similarly (23) becomes

$$\frac{\partial(\rho\,\overline{q})}{\partial t} + \nabla_h \cdot \overline{(\rho\,q\vec{\mathbf{V}}_{\mathbf{h}})} + \frac{\partial(\overline{q}\,\overline{m})}{\partial z} = M_c\frac{\partial\widetilde{q}}{\partial z}. \tag{32}$$

- For the detrainment layer ($\partial M_c/\partial z < 0$), we obtain from (29)

$$M_c\frac{\partial h_c}{\partial z} = 0, \tag{33}$$

$$\begin{aligned}
\rho\,L\,c - \frac{\partial}{\partial z}[M_c(s_c - \widetilde{s})] &= L\left[\frac{\partial M_c}{\partial z} q_c - \frac{\partial(M_c\,q_c)}{\partial z}\right] - \frac{\partial}{\partial z}\Big[M_c(s_c - \widetilde{s})\Big] \\
&= -LM_c\frac{\partial q_c}{\partial z} - M_c\frac{\partial s_c}{\partial z} + M_c\frac{\partial\widetilde{s}}{\partial z} - (s_c - \widetilde{s})\frac{\partial M_c}{\partial z} \\
&= \underbrace{-M_c\frac{\partial h_c}{\partial z}}_{equals\ 0} + M_c\frac{\partial\widetilde{s}}{\partial z} - (s_c - \widetilde{s})\frac{\partial M_c}{\partial z}
\end{aligned} \tag{34}$$

Therefore (22) becomes

$$\frac{\partial(\rho\,\overline{s})}{\partial t} + \nabla_h \cdot \overline{(\rho\, s\vec{\mathbf{V}}_{\mathbf{h}})} + \frac{\partial(\overline{s}\,\overline{m})}{\partial z} = M_c\frac{\partial\widetilde{s}}{\partial z} - (s_c - \widetilde{s})\frac{\partial M_c}{\partial z}. \tag{35}$$

Similarly (23) becomes

$$\frac{\partial(\rho\,\overline{q})}{\partial t} + \nabla_h \cdot \overline{(\rho\, q\vec{\mathbf{V}}_{\mathbf{h}})} + \frac{\partial(\overline{q}\,\overline{m})}{\partial z} = M_c\frac{\partial\widetilde{q}}{\partial z} - (q_c - \widetilde{q})\frac{\partial M_c}{\partial z}. \tag{36}$$

- For the entrainment layer,

(31) shows that the apparent heating (in the absence of radiative heating and evaporative cooling) is due to the *adiabatic (compression) warming by the downward motion between the clouds, which is compensating the cumulus mass flux* M_c ($M_c\,\partial\widetilde{s}/\partial z > 0$). (32) shows that the apparent moisture sink is due to the drying effect of the compensating subsidence ($M_c\,\partial\widetilde{q}/\partial z < 0$).

In addition, for the detrainment layer, (35) and (36) show that the detrainment of excess static energy, $s_c - \widetilde{s} = c_p(T_c - \widetilde{T})$, and excess moisture, $q_c - \widetilde{q}$, act to warm (if $T_c - \widetilde{T} > 0$) and moisten (if $q_c - \widetilde{q} > 0$) the environment.

In this simple case, the problem of parameterization of cumulus convection reduces to the *determination of* M_c *in terms of the large–scale variables.* Once M_c is known, we can obtain h_c from (26). Then using the relations

$$s_c - \widetilde{s} = c_p(T_c - \widetilde{T}) = \frac{1}{1+\gamma}(h_c - \widetilde{h}^*), \tag{37a}$$

$$q_c - \widetilde{q}^* = \frac{1}{1+\gamma}\,\frac{1}{L}(h_c - \widetilde{h}^*), \tag{37b}$$

where

$$\gamma \equiv \frac{L}{c_p}\Big(\frac{\partial\widetilde{q}^*}{\partial\widetilde{T}}\Big)_p \quad \text{(Arakawa, 1969)},$$

we can determine T_c and q_c. In the above $\widetilde{q}^*(\widetilde{T},p)$ is the saturation mixing ratio at $(\widetilde{T},p)$. The rate of condensation c is determined by (25), because we know M_c and q_c in addition to $\widetilde{q}\,(\approx\overline{q})$.