A Numerical Model for Simulating Convection

1   The model equations

You will develop a numerical model and use it to simulate a convective flow. The model is based on the quasi-compressible outflow model (QCOM) described in Droege-meier and Wilhelmson (1987). The model predicts the horizontal velocity (v), the vertical velocity (w), the potential temperature (θ), and the non-dimensional perturbation pressure (π_1). The compressible, non-rotating, adiabatic equations in Cartesian coordinates (y, z) are:

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial y} - w\frac{\partial v}{\partial z} - c_p\theta_0\frac{\partial \pi_1}{\partial y} + D_v, \tag{1}$$

$$\frac{\partial w}{\partial t} = -v\frac{\partial w}{\partial y} - w\frac{\partial w}{\partial z} - c_p\theta_0\frac{\partial \pi_1}{\partial z} + g(\frac{\theta}{\theta_0} - 1) + D_w, \tag{2}$$

$$\frac{\partial \theta}{\partial t} = -v\frac{\partial \theta}{\partial y} - w\frac{\partial \theta}{\partial z} + D_\theta, \tag{3}$$

$$\frac{\partial \pi_1}{\partial t} = -\frac{c_s^2}{c_p\theta_0^2}\left[\frac{\partial}{\partial y}(\theta_0 v) + \frac{\partial}{\partial z}(\theta_0 w)\right]. \tag{4}$$

See Klemp and Wilhelmson (1978) for a derivation of (4). In our version, we neglect the height variation of the density. In the equations above, $\pi = (p/p_r)^{R/c_p}$, where $p_r = 1000$ mb, R is the gas constant for dry air, and c_p is the specific heat capacity at constant pressure for dry air, and c_s is the constant speed of sound. The terms D_v, D_w, and D_θ represent turbulent mixing. Variables with a subscript 0 refer to the basic state, which varies with height only. A subscript 1 indicates the departure from the basic state. The basic state is in hydrostatic balance:

$$\frac{d\pi_0}{dz} = -\frac{g}{c_p\theta_0}. \tag{5}$$

2   Quasi-compressibility approximation

The quasi-compressibility approximation involves artificially slowing down the sound wave modes so that a larger time step may be used. For large c_s, the solution of the equations approaches that of the anelastic system. For small c_s (near the speed

of the fastest non-acoustic signal in the simulation), incorrect results occur due to artificially strong coupling between gravity wave and acoustic modes. Droegemeier and Wilhelmson found that for $c_s > 50$ m/s, elastic energy is much less than than the kinetic energy, and solutions are essentially independent of c_s.

3 Finite-difference equations

The grid is staggered, with π and θ located at the central point of the stencil, w one-half grid interval above and below the central point, and v one-half grid interval to the left and right of the central point. We will use the operator notation:

$$\delta_{n\xi}\phi(\xi) = \frac{\phi(\xi + n\Delta\xi/2) - \phi(\xi - n\Delta\xi/2)}{n\Delta\xi},$$

$$\overline{\phi(\xi)}^{n\xi} = \frac{\phi(\xi + n\Delta\xi/2) + \phi(\xi - n\Delta\xi/2)}{2},$$

where ϕ denotes a dependent variable, ξ is the independent variable, $\Delta\xi$ is a grid interval, and $n\Delta\xi$ is the interval over which the opertion takes place.

The right-hand sides of (1)-(4) will be denoted f_v, f_w, f_θ, and f_π. Their centered, second-order accurate finite-difference forms are

$$f_v = -v\delta_{2y}v - \overline{\overline{w}^y\delta_z v}^z - c_p\theta_0\delta_y\pi_1 + D_v, \tag{6}$$

$$f_w = -\overline{\overline{v}^z\delta_y w}^y - w\delta_{2z}w - c_p\overline{\theta_0}^z\delta_z\pi_1 + g\left(\frac{\overline{\theta}^z}{\overline{\theta_0}^z} - 1\right) + D_w, \tag{7}$$

$$f_\theta = -\overline{v\delta_y\theta}^y - \overline{w\delta_z\theta}^z + D_\theta, \tag{8}$$

$$f_\pi = -\frac{c_s^2}{c_p\theta_0^2}\left[\delta_y(\theta_0 v) + \delta_z(\overline{\theta_0}^z w)\right]. \tag{9}$$

Each of (1)-(4) can be written as

$$\frac{\partial\phi}{\partial t} = f_\phi,$$

which can be approximated with the second-order Adams-Bashforth scheme as

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{3}{2}f_\phi^n - \frac{1}{2}f_\phi^{n-1}.$$

Here Δt is the time step, and the superscripts indicate the time levels. Because the Adams-Bashforth scheme is a two-level scheme, it cannot be used for the first time step of a simulation. Instead, the forward scheme may be used:

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = f_\phi^n.$$

The time step is based on the CFL criterion for linear sound waves:

$$\Delta t < \frac{(\Delta y \Delta z)^{1/2}}{c_s}.$$

For example, if $\Delta y = \Delta z = 100$ m and $c_s = 100$ m/s, then $\Delta t < 1$ s for stability. Due to other aspects of the finite-difference scheme, the Δt that is actually used is usually smaller by a factor of $1/2$ to $1/4$.

4 Turbulence closure

For simplicity, we will use the eddy viscosity approach. Then terms D_v, D_w, and D_θ each have the form

$$K_\phi \nabla^2 \phi,$$

or, in finite-difference form,

$$K_\phi [\delta_y(\delta_y \phi) + \delta_z(\delta_z \phi)],$$

where K_ϕ is the eddy diffusivity.

5 Boundary conditions

The lower and upper boundaries (at $z = 0$ and $z = H$) are both rigid (i.e., $w = 0$), either free-slip ($\partial v/\partial z = 0$) or no-slip ($v = 0$) and either non-conducting (insulating, $\partial \theta/\partial z = 0$) or conducting ($\theta = \theta_0$). The lateral boundary conditions (at $y = 0$ and $y = L$) are cyclic (periodic, $\phi(y + L, z) = \phi(y, z)$).

We can implement these boundary conditions in a way that allows the same code to be used for the grid points adjacent to the boundaries as is used by the remainder of the grid points. We do this by adding an extra level of grid points a distance $\Delta z/2$ below the lower boundary and another extra level a distance $\Delta z/2$ above the upper boundary, and by adding an extra column of grid points a distance $\Delta y/2$ to the left

of the left boundary and another extra column of grid points a distance $\Delta y/2$ to the right of the right boundary.

Use j to denote a grid column at $y = y_j = j\Delta y$, and k a grid level at $z = z_k = k\Delta z$. At the left boundary, $j = 0$, while at the right boundary, $j = J$. Similarly, at the lower boundary, $k = 0$, while at the top boundary, $k = K$. The grid column at $y = \Delta y/2$ is at $j = 1/2$, while the extra grid column at $y = -\Delta y/2$ is at $j = -1/2$. A variable ϕ located at grid column j and grid level k is denoted $\phi_{j,k}$.

Free-slip: $\partial v/\partial z = 0$ at $z = 0, H$. Thus, $\delta_z v = 0$ at $z = 0, H$, and $v_{j,k+1/2} - v_{j,k-1/2} = 0$ for $k = 0, K$.

No-slip: $v = 0$ at $z = 0, H$. Thus, $\bar{v}^z = 0$ at $z = 0, H$, and $v_{j,k+1/2} + v_{j,k-1/2} = 0$ for $k = 0, K$.

Non-conducting: $\partial\theta/\partial z = 0$ at $z = 0, H$. Thus, $\delta_z\theta = 0$ at $z = 0, H$, and $\theta_{j,k+1/2} - \theta_{j,k-1/2} = 0$ for $k = 0, K$.

Conducting: $\theta = \theta_0$ at $z = 0, H$. Thus, $\bar{\theta}^z = \theta_0$ at $z = 0, H$, and $(\theta_{j,k+1/2} + \theta_{j,k-1/2})/2 = (\theta_0)_k$ for $k = 0, K$.

Cyclic: $\phi(y + L, z) = \phi(y, z)$. Thus, $\phi_{j+J,k} = \phi_{j,k}$. To obtain the values needed for the extra grid columns at $j = -1/2$ and $j = J+1/2$, apply the cyclic boundary condition for $j = -1/2$ and $j = 1/2$.

6   Programming notes

See http://www.met.utah.edu/class/skrueger/6150/qcom.htm for links to a Fortran program template file, a diagram of the arrangement of the model variables upon the grid, information about printing source code, the Fortran compiler and loader (f77), and the error utility; and subroutine PRINT, which you can use to easily print the results of your simulations.

Table 1: Array subscripting conventions.

$\phi_{j,k}$	FORTRAN array element
$w_{j,0}$	`w(j,0)`
$w_{j,1}$	`w(j,1)`
$\theta_{j,-1/2}$	`th(j,0)`
$\theta_{j,1/2}$	`th(j,1)`
$v_{0,k}$	`v(0,k)`
$v_{1,k}$	`v(1,k)`
$\theta_{-1/2,k}$	`th(0,k)`
$\theta_{1/2,k}$	`th(1,k)`

Table 2: Predicted variable ranges.

variable	subscript range	FORTRAN subscript range
w	j=1/2, J-1/2	`j=1, jt`
w	k=1, K-1	`k=1, kw`
v	j=1, J	`j=1, jv`
v	k=1/2, K-1/2	`k=1, kt`
θ	j=1/2, J-1/2	`j=1, jt`
θ	k=1/2, K-1/2	`k=1, kt`

Table 1 lists the locations of variables on the grid and the corresponding FOR-TRAN array elements. The variable π_1 is co-located with θ and follows the same conventions.

Table 2 lists the predicted variable ranges. These variables are calculated using the prognostic equations. (Boundary conditions are applied after predicting all of the variables in the ranges listed.) We specify `jt` and `kt`. Then, `kw = kt-1` and `jv = jt`.

7 Simulation 1: Conduction (Due February 26, 2009)

This case will correspond to the conduction regime. In this regime, there is no fluid motion, so only Eq. (3) is needed. The (potential) temperature will be specified to be steady and horizontally uniform at the upper and lower boundaries. Under these conditions, Eq. (3) simplifies to

$$\frac{\partial \theta}{\partial t} = D_\theta = K_\theta \frac{\partial^2 \theta}{\partial z^2}. \tag{10}$$

Analytically determine the general steady-state ($\partial \theta / \partial t = 0$) solution of (10) that satisfies the boundary conditions.
    The following parameters, along with $K_v = K_w = K_\theta$, correspond to Ra=500, which is subcritical for free-slip boundary conditions, when K_v, K_w, and K_θ are interpreted as molecular diffusivities: $H = 500$ m, $L = H$, $K_\theta = 100$ m^2 s^{-1}, $\theta_0(0) = 288$ K, $\theta_0(H) = \theta_0(0) - \Delta\theta$, $\Delta\theta = 1.2$ K. **Use your analytic, general, steady-state solution to calculate the specific solution for these parameters.**
    Now use your program to calculate a numerical steady-state solution for the same parameters. Start with $\theta(y, z) = (\theta_0(0) + \theta_0(H))/2$. Use jt=1 and kt=20. Use $\Delta t = 0.2(\Delta z)^2/K_\theta$, where Δt, Δz, and K_θ are all in MKS (meter-kilogram-second) units. Run for 200 s. **Plot $\theta(z)$ every 10 s on a single graph. Also plot the the corresponding analytic steady-state solution.**

8 References

Droegemeier, K. K., and R. B. Wilhelemson, 1987: Numerical simulation of thunderstorm outflow dynamics. Part I: Outflow sensitivity experiments and turbulence dynamics. *J. Atmos. Sci.,* **44,** 1180–1210.

Klemp, J. B., and R. B. Wilhelemson, 1978: The simulation of three-dimensional convective storm dynamics. *J. Atmos. Sci.,* **35,** 1070–1096.