Atmospheric Sciences 6150
Cloud System Modeling

(Revised 10 September 2019)

Plume Buoyancy Flux

Consider a point source of smoke. Let the smoke concentration (or density) be ϕ (kg/m^3) outside the source. The governing (conservation) equation is

$$\frac{d\phi}{dt} = s_\phi,$$

which can be written as

$$\frac{\partial\phi}{\partial t} = -\nabla\cdot(\mathbf{v}\phi) + s_\phi$$

where s_ϕ is the source strength per unit volume (kg/s/m^3).

Assume that the source is a *point* source. We want to determine the total or integrated source, S_ϕ (kg/s). Integrate over a control volume that encloses the source. If the volume is small enough,

$$\oint_V \partial\phi/\partial t\, dV \to 0.$$

Then, for this volume,

$$S_\phi \equiv \oint_V s_\phi\, dV = \oint_V \nabla\cdot(\mathbf{v}\phi)\, dV = \oint_S (\mathbf{v}\phi)\cdot dS \qquad (1)$$

which says that **the total source (kg/s) equals the flux through the surface (kg/m^2/s) enclosing the source, integrated over the surface (m^2).**

For a plume due to a point source of buoyancy at the surface, we need to know the total vertical flux of buoyancy flux due to the source. Eq. (1) can be applied to buoyancy, B, as well as to smoke:

$$S_B = \oint_S (\mathbf{v}B) \cdot dS \tag{2}$$

If we specify the control volume to be a cylinder, then (2) becomes

$$S_B = \int_0^{2\pi} \int_0^R \int_0^H (\mathbf{v}B) \cdot \hat{\mathbf{n}} \, dz \, r \, dr \, d\theta, \tag{3}$$

where r, z, and θ are the cylindrical coordinates: radius, height, and azimuth angle, and $\hat{\mathbf{n}}$ is the unit normal vector to the surface of the cylinder. There is no air flow through the bottom of the cylinder because the bottom is a rigid surface. The air that flows into the control volume through the sides contains no buoyancy. After air flows into the control volume, it gains buoyancy, and then flows out the top. Only the air that flows out the top contains buoyancy, therefore (3) becomes

$$S_B = 2\pi R^2 \, (\overline{wB})_H = A(\overline{wB})_H, \tag{4}$$

where A is the area of the top of the control volume, $(\overline{wB})_H$ is the average flux through this surface. By definition,

$$F_0 = A(\overline{wB})_H$$

is the total buoyancy flux at the source ($\mathrm{m}^4/\mathrm{s}^3$), but from (4) we also see that

$$F_0 = S_B \equiv \oint_V s_B \, dV. \tag{5}$$

Eq. (5) says that F_0**, the total (area-integrated) buoyancy flux at the source, equals** S_B**, the total source of buoyancy.** In other words, to determine F_0, we need to determine S_B.

Buoyancy is defined as

$$B \equiv \frac{g}{T_0}(T - T_0),$$

therefore

$$\frac{dB}{dt} = \frac{g}{T_0}\frac{dT}{dt}.$$

Let Q be the heating rate per unit mass of air (J/s/kg). Then

$$c_p \frac{dT}{dt} = Q$$

and

$$\frac{dB}{dt} = \frac{g}{T_0}\frac{Q}{c_p}.$$

Then

$$\begin{aligned} S_B &\equiv \oint_V s_B \, dV \\ &= \oint_V \frac{dB}{dt} \, dV \\ &= \oint_V \frac{g}{T_0}\frac{Q}{c_p} \, dV \\ &= \frac{g}{\rho c_p T_0} \oint_V \rho \, Q \, dV \\ &= \frac{g}{\rho c_p T_0} \hat{Q}, \end{aligned}$$

where

$$\hat{Q} \equiv \oint_V \rho \, Q \, dV$$

is the total (volume integrated) heating rate (J/s).