Scale Analysis of the Eqs. of Motion

(on a rotating earth)

Holton (2.8):

$$\frac{d\underset{\sim}{V}}{dt} = -2\underset{\sim}{\Omega} \times \underset{\sim}{V} - \frac{1}{\rho}\nabla p + \underset{\sim}{g} + \underset{\sim}{F}$$

coriolis force   p.g.f.   gravity   friction

The component eqs. in spherical coordinates are derived in Holton §2.3, see (2.19)–(2.21). These are listed in Tables 2.1 and 2.2 (handout).

Scale analysis involves determining whether some terms are negligible for the motions of interest.

To simplify the eqs., we use observations to assign values to characteristic scales.

Scale analysis of the eqs. of motion for synoptic-scale motions is done in §2.4.

For synoptic-scale motions:

horizontal velocity scale	U	$\sim$	10 m/s
vertical velocity scale	W	$\sim$	1 cm/s
length scale	L	$\sim$	10^6 m
depth scale	H	$\sim$	10^4 m
horiz. pressure fluctuation scale ($\Delta P \sim 10$ mb $= 10^3$ Pa)	$\frac{\delta P}{\rho}$	$\sim$	10^3 m²/s²
(advective) time scale	$\frac{L}{U}$	$\sim$	10^5 s

Parameters that don't depend on the type of motion:

$f_0 \approx 10^{-4}\ s^{-1}$ (mid-latitude)

$a \approx 6 \times 10^6\ m \sim 10^7\ m$ (earth's radius)

$\rho \approx 1\ kg/m^3$ (sea level)

$P_0 = 10^5\ Pa$ (sea level)

$g = 10\ m/s^2$

$\nu \approx 10^{-5}\ m^2/s$ (kinematic viscosity)

$U \sim 10\ \mathrm{m\,s^{-1}}$	horizontal velocity scale
$W \sim 1\ \mathrm{cm\,s^{-1}}$	vertical velocity scale
$L \sim 10^6\ \mathrm{m}$	length scale [$\sim 1/(2\pi)$ wavelength]
$H \sim 10^4\ \mathrm{m}$	depth scale
$\delta P/\rho \sim 10^3\ \mathrm{m^2\,s^{-2}}$	horizontal pressure fluctuation scale
$L/U \sim 10^5\ \mathrm{s}$	time scale

Table 2.1 *Scale Analysis of the Horizontal Momentum Equations*

	A	B	C	D	E	F	G
x-Eq.	$\frac{Du}{Dt}$	$-2\Omega v \sin\phi$	$+2\Omega w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
y-Eq.	$\frac{Dv}{Dt}$	$+2\Omega u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2\tan\phi}{a}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
$(\mathrm{m\,s^{-2}})$	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}

Table 2.2 *Scale Analysis of the Vertical Momentum Equation*

z-Eq.	Dw/Dt	$-2\Omega u\cos\phi$	$-(u^2+v^2)/a$	$=-\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	$f_0 U$	U^2/a	$P_0/(\rho H)$	g	$\nu W H^{-2}$
$\mathrm{m\,s^{-2}}$	10^{-7}	10^{-3}	10^{-5}	10	10	10^{-15}

Eq. of motion (on a rotating earth)

$$\frac{d\underset{\sim}{V}}{dt} = -2\underset{\sim}{\Omega} \times \underset{\sim}{V} - \frac{1}{\rho}\nabla p + \underset{\sim}{g} + \underset{\sim}{F}$$

$\uparrow$ effective gravity $\uparrow$ friction force

Scale analysis

Convective clouds:

$U \sim 10^1$ m/s	$f_0 \simeq 10^{-4}$ s^{-1}
$W \sim 10$ m/s	$a \approx 6 \times 10^6$ m
$L \sim 10^4$ m	$\rho \approx 1$ kg/m^3
$D \sim 10^4$ m	$P_0 = 10^5$ Pa
$\Delta P \sim 1$ mb $= 100$ Pa	$g = 10$ m/s^2
$L/U \sim \frac{10^4}{10} = 10^3$ s	$\frac{\rho'}{\rho_0} \sim \frac{T'}{T_0} \sim 10^{-2}$

Horiz. mom. Eqs, (Table 2.1, Holton)

A	B	C	D	E	F	
$\frac{U^2}{L}$	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\Delta P}{\rho L}$	
10^{-2}	10^{-3}	10^{-3}	10^{-5}	10^{-5}	10^{-2}	m/s^2

Geostrophic balance does not occur;
accel. balanced by p.g.f.
Coriolis force is not important.

Cu		
10	$U \sim 10\ \mathrm{m\ s^{-1}}$	horizontal velocity scale
10 m/s	$W \sim 1\ \mathrm{cm\ s^{-1}}$	vertical velocity scale
10^4	$L \sim 10^6\ \mathrm{m}$	length scale $[\sim 1/(2\pi)$ wavelength$]$
10^4	$H \sim 10^4\ \mathrm{m}$	depth scale
10^2	$\delta P/\rho \sim 10^3\ \mathrm{m^2\ s^{-2}}$	horizontal pressure fluctuation scale
10^3	$L/U \sim 10^5\ \mathrm{s}$	time scale

Table 2.1 *Scale Analysis of the Horizontal Momentum Equations*

	A	B	C	D	E	F	G
x-Eq.	$\frac{Du}{Dt}$	$-2\Omega v \sin\phi$	$+2\Omega w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv \tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial x}$	$+F_{rx}$
y-Eq.	$\frac{Dv}{Dt}$	$+2\Omega u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2 \tan\phi}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial y}$	$+F_{ry}$
Scales	U^2/L	$f_0 U$	$f_0 W$	$\frac{UW}{a}$	$\frac{U^2}{a}$	$\frac{\delta P}{\rho L}$	$\frac{\nu U}{H^2}$
$(\mathrm{m\ s^{-2}})$	10^{-4}	10^{-3}	10^{-6}	10^{-8}	10^{-5}	10^{-3}	10^{-12}
Cu	10^{-2}	10^{-3}	10^{-3}	10^{-5}	10^{-6}	10^{-2}	10^{-12}

Table 2.2 *Scale Analysis of the Vertical Momentum Equation*

z-Eq.	Dw/Dt	$-2\Omega u \cos\phi$	$-(u^2+v^2)/a$	$= -\rho^{-1}\partial p/\partial z$	$-g$	$+F_{rz}$
Scales	UW/L	$f_0 U$	U^2/a	$P_0/(\rho H)$	g	$\nu W H^{-2}$
$\mathrm{m\ s^{-2}}$	10^{-7}	10^{-3}	10^{-5}	(10) 10^{-1}	(10) 10^{-1}	10^{-15}
Cu	10^{-2}	10^{-3}	10^{-5}	(10) 10^{-2}	(10) 10^{-1}	10^{-12}

$$R_o \equiv \frac{U}{f_o L} = \frac{10}{10^{-4}/10^{4}} = 10 \; ;$$ smallness indicates validity of geostrophic approx.

$R_o = 0.1$ for synoptic scale.

---

vert. mom. Eq. (Table 2.2, Holton)

$\frac{dw}{dt}$	$-2\Omega u \cos\phi$	$-\frac{u^2+v^2}{a}$	$= -\frac{1}{\rho}\frac{\partial p}{\partial z}$	
$\frac{UW}{L}$	$f_o U$	$\frac{U^2}{a}$	$\frac{P_o}{\rho D}$	
10^{-2}	10^{-3}	10^{-5}	10^{1}	10^{1}

[Hydrostatic balance exists to a high degree.]

only part of p that varies horiz. is directly coupled to velocity field, so we must use this and horizontally-varying density field to check scaling.

We decompose p and ρ so

$$p(x,y,z,t) = p_o(z) + p'(x,y,z,t)$$
$$\rho(\quad " \quad) = \rho_o(z) + \rho'(\quad " \quad)$$

where p_o, ρ_o exactly satisfy

$$-\frac{1}{\rho_o}\frac{dp_o}{dz} = g.$$

For atmos. at rest, $p' = \rho' = 0$.

we find, assuming $\rho'/\rho_0 \ll 1$, $p'/p_0 \ll 1$, that

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{(\rho_0+\rho')}\frac{\partial}{\partial z}(p_0+p') - g$$

$$= \frac{-1}{(\rho_0+\rho')}\left[\frac{\rho_0-\rho'}{(\rho_0-\rho')}\right]\frac{\partial}{\partial z}(p_0+p') - g \qquad \times \frac{\rho_0-\rho'}{\rho_0-\rho'}$$

$$\simeq -\frac{\rho_0-\rho'}{[\rho_0^2]}\frac{\partial}{\partial z}(p_0+p') - g \qquad (\rho_0+\rho')(\rho_0-\rho_0) = \rho_0^2 - \rho'^2 \approx \rho_0^2$$

$$= -\left[\left(\frac{1}{\rho_0} - \frac{\rho'}{\rho_0^2}\right)\right]\frac{\partial}{\partial z}(p_0+p') - g \qquad \text{separate terms}$$

$$= -\frac{1}{\rho_0}\frac{dp_0}{dz} + \frac{\rho'}{\rho_0^2}\frac{dp_0}{dz} - \frac{1}{\rho_0}\left[\frac{\partial p'}{\partial z} - \overbrace{\frac{\rho'}{\rho_0}\frac{\partial p'}{\partial z}}^{\text{neglect}}\right] - g \qquad \text{Multiply}$$

cancel (hydrostatic eq.)

$$\simeq \underbrace{-\frac{\rho'}{\rho_0}g}_{\text{buoyancy force}} - \underbrace{\frac{1}{\rho_0}\frac{\partial p'}{\partial z}}_{\text{perturbation (non-hydrostatic) p.g.f.}}$$

use $\frac{dp_0}{dz} = -\rho_0 g$ (2,28)

Now check scaling:

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho_0}\left[\rho' g + \frac{\partial p'}{\partial z}\right] \qquad (2,28)$$

These terms have magnitudes

	$\rho'/\rho\, g$	$\Delta P/\rho D$	
	10^{-1}	10^{-2}	(cu)
	10^{-1}	10^{-1}	(synoptic)

[These are $\simeq$ accel. term, so vert. accel. isn't negligible. Coriolis term is small.]

TYPICAL VALUES OF DIVERGENCE, VERTICAL MOTION AS FUNCTION OF SCALE

	Horiz. Scale	DU /	DX	Horiz.. Diverg.	w (1-5 km)	Rain Rate (mm/h)	Time Scale
PLANETARY	10^4 km	5 m/s	5000 km	10^{-6}/s	1-5 mm/s	n/a	10^6 s (week)
SYNOPTIC	10^3 km	5 m/s	500 km	10^{-5}/s	1-5 cm/s	0.2-1	10^5 s (day)
MESO	10^2 km	5 m/s	50 km	10^{-4}/s	10-50 cm/s	2-10	10^4 s (3 h)
CONVECTIVE	10^1 km	5 m/s	5 km	10^{-3}/s	1-5 m/s	20-100	10^3 s (1/4 h)
"SUPERCELL"	[10^0 km	5 m/s	0.5 km]	10^{-2}/s	10-50 m/s	n/a	10^2 s (2 min)