An Explication of the Coriolis Effect

Norman A. Phillips
Merrimack, New Hampshire

ABSTRACT

Inertia circle motion is a familiar concept demonstrating the behavior of a horizontally moving particle on the rotating earth in response to the Coriolis force. Three dynamical principles valid in nonrotating space are used to explain the principal properties of this motion without the need for solving differential equations. The motion is an oscillation about a central latitude selected by the particle's angular momentum, in which the restoring force is set by the relative magnitudes of the poleward attraction of gravity and the equatorward attraction of the centrifugal force as the particle oscillates about the central latitude.

1. Goal and principles

The object of this note is to provide an interpretation of how motion on the rotating earth is affected by that rotation. The physical principles to be used are extremely well known, but as far as I know, have not been applied to this problem in precisely this style, a style designed to avoid the solution of differential equations. In spite of its long history, the subject is very much alive, especially in a pedagogical context (Stommel and Moore 1989; Duran 1993). Persson (1998) provides many references and a historical perspective. Two examples are discussed here, *inertia circle* motion and *geostrophic flow*. It is suggested that the results might provide a simple physical understanding to supplement the elementary manipulations in the conventional derivation of the Coriolis force, which, according to Persson (1998) and Stommel and Moore (1989), remain a mystery to many.

Inertia circle motion is the idealized behavior of a particle that is confined to horizontal motion on the earth and, when viewed in a framework fixed to the rotating earth, has its horizontal displacements subject only to the Coriolis force. This makes it a simple example from which to understand the Coriolis effect on motion.

For horizontal motion, that force is equal to the product of the speed, C, and the Coriolis parameter, f:

$$\text{Coriolis force per unit mass} = fC,\ f = 2\Omega\sin\phi. \quad (1)$$

Here Ω is the angular velocity of the earth and ϕ is latitude. The force is directed 90° to the right of the motion in the Northern Hemisphere and 90° to the left in the Southern Hemisphere. It therefore does not change the speed. In the absence of other forces, it gives rise to clockwise circular motion in the Northern Hemisphere and counterclockwise motion in the Southern Hemisphere. (For brevity, further discussion will focus on the Northern Hemisphere.)

The circular motion requires an inward acceleration equal to C^2 divided by the radius. Equating this to fC from (1), we find that the radius of the inertia circle is

$$C/f. \quad (2)$$

This solution is idealized by the convenience of considering only small changes in latitude. A similar assumption will be taken for granted in the remainder of this note. Ripa (1997) has carefully considered the

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Corresponding author address: Norman A. Phillips, 18 Edward Lane, Merrimack, NH 03054.
E-mail: NapMar18@aol.com
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effect of variations in latitude and approximate ways to allow for this element.

The above is the explanation of an inertia circle from the viewpoint of an observer in the rotating system who is familiar with the Coriolis force. But physical principles are often most familiar when viewed from a coordinate system fixed in space. We will therefore use a *nonrotating* coordinate system for a view of the physical processes underlying this simplest example of the Coriolis force.

In a nonrotating coordinate system, however, it is necessary to recognize two related effects that do not appear explicitly when the equations of motion are expressed relative to the rotating earth. As a result of its rotation, the earth is an oblate spheroid, such that gravitational attraction at the earth's surface has a horizontal component directed toward the pole. This component, F_G, equals the horizontal component of the earth's centripetal acceleration:

$$F_G = \Omega^2 R \sin\phi, \tag{3}$$

where R is the distance to the axis of the earth. (The R at the surface of the earth is closely equal to the radius of the earth times the cosine of the latitude.) This component of gravitational attraction is what keeps everything from sliding to the equator, as shown in Fig. 1. Both effects are contained in the meridional component of the horizontal equations of motion when expressed in nonrotating spherical coordinates:

$$dV/dt + (U^2/R)\sin\phi = F_G. \tag{4}$$

Here V is the northward component of the horizontal velocity, U is the absolute eastward component of velocity, and t is time. For a particle at rest on the earth, U equals the product ΩR. For clarity, the expression $+ (U^2/R)\sin\phi$ will be referred to as the centripetal acceleration. It equals F_G when $U = \Omega R$.

The other horizontal equation of motion can be written

$$dU/dt - (UV/R)\sin\phi = 0. \tag{5}$$

Since there is no force in the zonal direction, and $V \sin\phi$ is equal to $-\mathrm{d}R/\mathrm{d}t$, (5) can be manipulated to show that the angular momentum of a particle is not changed:

$$\text{angular momentum} = RU = R(u + \Omega R) = \text{constant}. \tag{6}$$

Here u is the eastward velocity relative to the rotating earth.

Durran (1993) has shown how F_G gives rise to inertia circle motion. His demonstration is based on solving the equations of motion for a parcel in nonrotating coordinates. Like Durran, I present my arguments in the context of nonrotating coordinates. However, the major aspects of inertia circle motion can be obtained without solving differential equations. This will expose the dynamical principles that are involved. Three familiar principles will be sufficient.

1) *Conservation of angular momentum.* From (6), we see that this determines the change of U as a particle changes its distance from the rotation axis.
2) *The energy principle:*

$$\text{change in kinetic energy} = \text{force} \times \text{displacement}. \tag{7}$$

This will determine the speed of the particle.

3) *The meridional equation of motion,* (4). This determines changes in the poleward velocity of the parcel. *Only the sign of dV/dt is necessary in the following discussion.* A slightly more detailed ex-

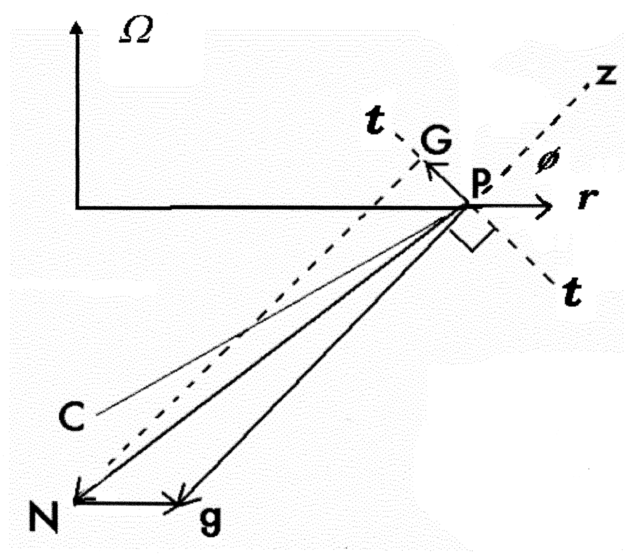

FIG. 1. Orientation of forces at a point P on the surface of the earth. Here, tt is a tangent plane. PC points to the center of the earth. **Pg** is local gravity and defines the local vertical, perpendicular to the tangent plane. **Pg** is gravity, the sum of **PN**, the Newtonian gravitation vector, and **Ng** or **Pr**, the centrifugal force. **PG** is the horizontal component of the gravitation vector **PN**, given by projection of **PN** on the tangent plane. **PG** is equal and opposite to the projection of **Pr** on the tangent plane. ϕ is the geodetic latitude, the angle between the equatorial plane and the local vertical.

amination of (4) will be needed in the analysis of geostrophic flow, however.

Principles 1 and 2 are specialized integral statements of the equations of motion that are valid for inertial motion. Stommel and Moore have formulated two constraints governing inertial motion on a rotating equipotential surface (1989, 245–246.) Principle 1 is one of their constraints. The other is similar to principle 2, but in the more elegant form of conservation of the kinetic energy of the absolute motion and the potential energy associated with F_G. This yields a simple deduction of the extrema of R. The form 2 is used here instead because it focuses more explicitly on the role of F_G and because it leads to evaluation of the relative velocities. Principle 3 is necessary to examine the interplay of F_G with the centripetal acceleration that determines the sign of dV/dt.

2. A northward-moving particle

We begin by observing a parcel as it crosses latitude ϕ_0 at radius R_0 in a northward direction with a purely meridional velocity,

$$V_0 > 0.$$

At this initial point, identified by subscript zero, the relative motion is

$$u_0 = 0, \qquad v_0 = V_0 > 0,$$

and its constant absolute angular momentum is

$$\mathrm{AM} = U_0 R_0.$$

The detailed arguments proceed as follows.

1) Since u is zero, the particle's absolute zonal velocity U is equal to $R\Omega$ and its centripetal acceleration is equal to F_G. According to principle 3, dV/dt is zero.
2) But as the inertia of the particle carries it to higher latitudes, conservation of angular momentum according to principle 1 will assign it an eastward velocity component:

$$u = [(R_0 + R)/R]\Omega(R_0 - R)$$
$$= [(R_0 + R)/R]\Omega\sin\phi\Delta s. \qquad (8a)$$

Here $\Delta s = -\Delta R/\sin\phi$ is meridional arc length along the earth's surface.
3) For small $(R_0 - R)/R$ the bracket term is approximately equal to 2. Using the value of f defined in (1), we can write for later reference,

$$du = f\Delta s. \qquad (8b)$$

4) This increases the total eastward velocity $U = \mathrm{AM}/R$, and hence the U^2 term in principle 3. This in turn makes dV/dt negative.
5) But the particle will continue moving poleward until the negative dV/dt has reduced V to zero. How far will it move, using arguments available to an observer in nonrotating space? Principle 2 provides the answer. The initial kinetic energy is

$$\mathrm{KE}_0 = (1/2)[v_0^2 + (\Omega R_0)^2]. \qquad (9)$$

The kinetic energy when the parcel reaches its polemost location at latitude ϕ_1 and $R = R_1$, with zero meridional motion, is

$$\mathrm{KE}_1 = (1/2)U_1^2 = (1/2)(u_1 + \Omega R_1)^2. \qquad (10)$$

Here u_1 is given by (8a) with $R = R_1$. The difference must be due to the work done by the force (per unit mass) times the displacement. In differential form this is

$$d\mathrm{Work} = (\Omega^2 R\sin\phi) \times \Delta s = -(1/2)\,\Omega^2 dR^2. \quad (11)$$

6) Evaluation of the terms in (9) and (10), and integrating (11) from R_0 to R_1, shows that this balance reduces to the simple result

$$u_1^2 = v_0^2. \qquad (12)$$

This equality is consistent with the uniform speed of inertia circle motion found to exist in the relative system.[1]

Furthermore, reference to (8b) and (2) shows that the northward displacement is equal to the radius of an inertia circle:

$$\Delta s = u_1/f = v_0/f. \qquad (13)$$

[1]Conservation of the kinetic energy of relative motion is characteristic of inertial motion in general.

3. Return to the initial latitude

Returning now to our nonrotating framework, we consider the parcel at its new latitude ϕ_1, with zonal relative velocity u_1 and zero meridional velocity, $u_1 = v_0$, and $v_1 = 0$.

1) Its angular momentum is

$$\mathrm{AM} = U_1 R_1. \tag{14}$$

2) But the total zonal motion is still too large for $(U^2/R)\sin\phi$ to be balanced by F_G at that latitude, and the parcel now begins to move equatorward in obedience to principle 3.
3) Its eastward relative zonal velocity will now decrease according to principle 1. Eventually it will reach a lower latitude where u vanishes.
4) This latitude, ϕ_2, is determined again by principle 1:

$$\mathrm{AM} = U_2 R_2 = \Omega R_2^2 = U_0 R_0 = \Omega R_0^2.$$

We are back at the initial latitude ϕ_0.
5) A calculation of the work and kinetic energy changes in going from ϕ_1 to ϕ_2 is similar to that for the progression from ϕ_0 to ϕ_1, except that the work is now negative, and the equatorward motion means that v_2 is negative. The result is

$$u_2 = 0, \qquad v_2 = -u_1 = -v_0.$$

This is as it should be, since if we had been watching the particle in the Northern Hemisphere from the rotating frame starting with relative velocity u_1 to the east, it would begin to describe a clockwise inertia circle, turning equatorward. When it is one-quarter of the way around, it will have zero zonal relative motion and will have progressed equatorward a distance equal to the radius of an inertia circle.

The further progress of the particle below latitude ϕ_0 will, to the extent we ignore major changes in latitude, be a mirror image of what has happened in the progression from ϕ_0 through ϕ_1 to ϕ_2.

This reasoning has shown the particle to have the same speed at four locations in its return to the initial latitude. During this motion it has traveled northward a distance equal to the radius of an inertia circle and it has traveled southward a like amount. It is a reasonable inference that the motion has a constant speed of relative motion in a circle of radius C/f on the earth. The principles 1, 2, and 3 have each contributed to this behavior, behavior that in the rotating system of coordinates is due solely to the Coriolis force.[2]

4. Geostrophic motion

The most pervasive influence of the Coriolis force for the atmosphere and ocean is not encountered in inertia circle motion, but in the relatively steady flow that is characterized by a balance, in the rotating coordinate system, between the horizontal pressure gradient force and the Coriolis force. This is referred to as geostrophic motion, and it too can be explicated with the principles developed above for inertia flow.

Equation (8b) describes the zonal velocity u that is generated by conservation of angular momentum for a particle as it moves northward. Division of (8b) on each side by a time interval $\mathrm{d}t$ produces the equation

$$du/dt = f\,\Delta s/\mathrm{d}t = fv. \tag{15}$$

This shows that the particle could continue to move directly northward, that is, with u always zero, if an additional force (per unit mass), F_W, of magnitude fv were applied toward the west:

$$F_W = -fv. \tag{16}$$

This is the geostrophic relation for steady northward flow as expressed in the rotating coordinate system. *Angular momentum is no longer conserved, however.*

It remains to examine how steady zonal motion that is faster or slower than the earth can be maintained. From the inertia circle discussion, we saw that when the particle had reached latitude ϕ_1 with relative zonal velocity u_1, its zonal velocity U_1 was too large for cancellation in the meridional equation of motion (4) between F_G and the centripetal term $(U^2/R)\sin\phi$; whereupon the particle had a negative value for dV/dt.

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[2]Readers who like to see differential equations in spite of the avowed goal of this note can take comfort in the following. The meridional equation of motion (4), with $V = \mathrm{d}s/\mathrm{d}t$, s being surface distance in the northward meridional direction measured from latitude $\phi 0$, can be rewritten with the aid of (8a) and the decomposition preceding (18), in the standard form of the simple oscillator equation, $\mathrm{d}^2 s/\mathrm{d}t^2 + f^2 s = 0$.

This negative dV/dt can be cancelled by applying a suitable northward-directed force. The size needed to balance the extra angular velocity is determined by expanding the centripetal acceleration term in (4):

$$(U^2/R)\sin\phi = [(\Omega R + u)^2/R]\sin\phi = \Omega^2 R\sin\phi + 2\Omega u\ \sin\phi + u^2\sin\phi/R. \qquad (17)$$

The Ω^2 term is needed to cancel the gravitational force F_G. It is the second term that primarily determines the excess centripetal acceleration; the necessary northward force is then

$$F_N = +2\Omega u\sin\phi = fu. \qquad (18)$$

Equations (16) and (18) show that the rotation-induced accelerations of a moving particle on the rotating earth can be canceled by applying a force per unit mass of magnitude $f\ \times$ speed. The force should be applied toward the left of the motion in the Northern Hemisphere and toward the right in the Southern Hemisphere. When the force is supplied by a pressure gradient force per unit mass, this is referred to as geostrophic flow.

It might appear peculiar that different approaches have been taken to explain geostrophic balance in the zonal and meridional directions; conservation of angular momentum led to (8b) for u, whereas the geostrophic equation for v came from the expansion of the centripetal acceleration in (4). However, the conservation of angular momentum principle is derived from the zonal equation of motion (5), which treats the balance of zonal forces, which include F_W. Similarly, the derivation of the geostrophic relation between u and F_N is obtained from expanding the centripetal acceleration in (4), which describes the balance of meridional forces, which include F_N. Viewed at this deeper level, there is no peculiarity.

5. Conclusions

The analysis in earlier sections provides the following picture of inertial circle motion from the standpoint of a nonrotating coordinate system. It is a stable oscillation of a particle about a central latitude on the surface of the ellipsoidal rotating earth, in which there is a contest between the equatorward centrifugal force and the poleward attraction of gravity. The central latitude is defined by the angular momentum possessed by the particle. Poleward of the central latitude the angular momentum of the particle prescribes an eastward zonal velocity greater than that of the earth, which allows the centrifugal force to overbalance the poleward gravitational attraction. Conversely, when the particle is on the equatorial side of the central latitude, the zonal velocity is slower than that of the earth and the gravitational attraction exceeds the centrifugal force. This force imbalance accelerates the particle poleward.

The presence of a horizontal pressure force can cancel the acceleration found in inertial motion and allow motion to occur in a straight horizontal path on the earth. Angular momentum is no longer conserved in this situation.

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