

**Diffusional Growth**

Generally air saturated with respect to water is supersaturated with respect to ice. This sets up a vapor pressure gradient such that in mixed-phased clouds, ice grows at the expense of water.

As before if the ice crystal is spherical

\[
\frac{dm}{dt} = 4\pi r D (\rho_{v\infty} - \rho_{vr})
\]

But as we know, ice crystals can be some truly crazy shapes. We can address this problem by drawing an analogy between the flux of vapor in and out of a crystal with charge leakage from a capacitor of the same size and shape.

The capacitance \( C \) of a spherical capacitor is

\[
C = 4\pi r \epsilon_0
\]

where \( \epsilon_0 = 8.85 \times 10^{-12} C/N/m^2 \) is the permittivity of free space. Forgetting about \( \epsilon_0 \), in general

\[
\frac{dm}{dt} = C D (\rho_{v\infty} - \rho_{vC})
\]

where \( \rho_{vC} \) is the water vapor density at the surface. Following the same arguments as described for water droplets

\[
\frac{dm}{dt} = C G_i S_i
\]

where \( G_i = D \rho_v (\infty) \) and \( S_i = \frac{e(\infty) - e_{ati}}{e_{ati}} \).

Wallace and Hobbs outlines in Table 6.1 the types of growth that dominate. In some regimes it is the basal face that dominates and we get columnar shapes. In others, it is the prism face, and we get more plate like structures. In general, the greater the supersaturation with respect to ice, the more complicated the shape.

Looking at diffusional growth more closely, consider Wulff’s theorem, which is that for slow diffusional growth of ice crystals

\[
\frac{h_b}{h_p} = \frac{\sigma_b}{\sigma_p} \simeq 0.92
\]

where \( \sigma \) is the surface tension of the ice crystal basal (b) or prism (p) face, and \( h \) is the height of the face. To understand this better consider the Kelvin equation we came up with, which is that

\[
e^{\text{sat}} \propto \exp(k\sigma)
\]

so that the greater the surface tension the greater the saturation vapor pressure. This means that if we have a given vapor pressure \( e \), then the speed of condensation will go as

\[
\frac{dm}{dt} \propto e - e^{\text{sat}}
\]

Therefore if the surface tension of the basal facet is higher compared to the prism face, then saturation vapor pressure is higher relative to the basal face, and the vapor pressure gradient \( e - e^{\text{sat}} \) is lower. This means that if \( \sigma_p \) is higher than \( \sigma_b \), it
follows that vapor will preferentially condense on the basal face: simply it is harder to condense on the prism face due to the higher surface tension there. It’s a little counter-intuitive (just draw it to convince yourself), but the faster the basal face grows, the large $h_p$ becomes, explaining Wulff’s theorem. If $\sigma_p > \sigma_b$ then $h_p > h_b$. Wulff’s theorem explains the shapes of ice crystals well when supersaturations are very small and temperatures between -10 C and -22 C.

The problem is that Wulff’s theorem doesn’t take us particularly far in explaining the wide variety of crystal aspect ratios that are observed at other temperatures and supersaturations. Growth is not necessarily very slow, extremely thin liquid layers can form on ice during the growth process, and ice crystals do not grow through simple vapor deposition but rather in steps and spiral formations. The true physics that controls ice crystal growth largely remains a mystery.